Thm 2.3 (Chain rule): Suppose $U \subset R^m$ is open and $f : U \to V \subset \mathbf{R}^m$, $g : V \to \mathbf{R}^p$. Let $h = g \circ f$. Suppose f is differentiable at $a \in U$ and g is differentiable at $f(a) \in V$. Then h is differentiable at $a \in U$ and $D(h)_a = D(G)_{f(a)} D(f)_a$.

Let $R_h(\mathbf{x}, \mathbf{a}) = \frac{g(f(\mathbf{x})) - g(f(\mathbf{a})) - D(G)_{f(a)} D(f)_a (\mathbf{x}-\mathbf{a})}{||\mathbf{x}-\mathbf{a}||}$

Let $\mathbf{y} = f(\mathbf{x}), \mathbf{b} = f(\mathbf{a})$

$R_g(\mathbf{y}, \mathbf{b}) = \frac{g(\mathbf{y}) - g(\mathbf{b}) - D(G)_{\mathbf{b}}(\mathbf{y}-\mathbf{b})}{||\mathbf{y}-\mathbf{b}||}$ where $lim_{\mathbf{x}\to\mathbf{a}} R_g(\mathbf{y}, \mathbf{b}) = 0$

$R_f(\mathbf{x}, \mathbf{a}) = \frac{f(\mathbf{x}) - f(\mathbf{a}) - D(f)_a(\mathbf{x}-\mathbf{a})}{||\mathbf{x}-\mathbf{a}||}$ where $lim_{\mathbf{x}\to\mathbf{a}} R_f(\mathbf{x}, \mathbf{a}) = 0$

$\mathbf{y} - \mathbf{b} = f(\mathbf{x}) - f(\mathbf{a}) = D(f)_a(\mathbf{x}-\mathbf{a}) + ||\mathbf{x}-\mathbf{a}||R_f(\mathbf{x}, \mathbf{a})$

$R_g(\mathbf{y}, \mathbf{b}) = \frac{g(f(\mathbf{x})) - g(f(\mathbf{a})) - D(G)_{\mathbf{b}}[D(f)_a(\mathbf{x}-\mathbf{a}) + ||\mathbf{x}-\mathbf{a}||R_f(\mathbf{x},\mathbf{a})]}{||\mathbf{y}-\mathbf{b}||}$

$\frac{||\mathbf{y}-\mathbf{b}||R_g(\mathbf{y},\mathbf{b})}{||\mathbf{x}-\mathbf{a}||} = \frac{g(f(\mathbf{x})) - g(f(\mathbf{a})) - D(G)_{\mathbf{b}} D(f)_a(\mathbf{x}-\mathbf{a}) - D(G)_{\mathbf{b}}||\mathbf{x}-\mathbf{a}||R_f(\mathbf{x},\mathbf{a})}{||\mathbf{x}-\mathbf{a}||}$

$\frac{||\mathbf{y}-\mathbf{b}||R_g(\mathbf{y},\mathbf{b}) + D(G)_{\mathbf{b}}||\mathbf{x}-\mathbf{a}||R_f(\mathbf{x},\mathbf{a})}{||\mathbf{x}-\mathbf{a}||} = \frac{g(f(\mathbf{x})) - g(f(\mathbf{a})) - D(G)_{\mathbf{b}} D(f)_a(\mathbf{x}-\mathbf{a})}{||\mathbf{x}-\mathbf{a}||}$

$R_h(\mathbf{x}, \mathbf{a}) = \frac{||\mathbf{y}-\mathbf{b}||R_g(\mathbf{y},\mathbf{b}) + D(G)_{\mathbf{b}}||\mathbf{x}-\mathbf{a}||R_f(\mathbf{x},\mathbf{a})}{||\mathbf{x}-\mathbf{a}||}$

$R_h(\mathbf{x}, \mathbf{a}) = \frac{||f(\mathbf{x}) - f(\mathbf{a})||R_g(\mathbf{y},\mathbf{b})}{||\mathbf{x}-\mathbf{a}||} + D(G)_{\mathbf{b}} R_f(\mathbf{x}, \mathbf{a})$

$R_h(\mathbf{x}, \mathbf{a}) = \frac{||D(f)_a(\mathbf{x}-\mathbf{a}) + ||\mathbf{x}-\mathbf{a}||R_f(\mathbf{x},\mathbf{a})||R_g(\mathbf{y},\mathbf{b})}{||\mathbf{x}-\mathbf{a}||} + D(G)_{\mathbf{b}} R_f(\mathbf{x}, \mathbf{a})$

Cor 2.4: If $f, g \in C^r$ on U, V respectively, then $h = g \circ f \in \mathbf{C}^r$.

Proof by induction:

r = 1: Suppose $f, g \in C^1$ on U, V respectively.

Then $\frac{\partial f}{\partial x_i}$ exists and is continuous on U

Then $\frac{\partial g}{\partial x_i}$ exists and is continuous on V.

By Thm 1.3, f, g are differentiable on U, V respectively.

Hence by Thm 1.1, f is continuous. By Thm 2.3 $h = g \circ f$ is differentiable.

Thus by Thm 1.1, $\frac{\partial h}{\partial x_i}$ exist.

By Thm 2.3 it's Jacobian is $D(h)_x = D(g)_{f(x)} D(f)_x$.

Since $\frac{\partial f_i}{\partial x_j}$ are continuous on U for all i, j, each entry of $D(f)_x = (\frac{\partial f_i}{\partial x_j})$ is continuous

Since $\frac{\partial g_i}{\partial x_j}$ are continuous on V for all i, j and f is continuous, each entry of $D(g)_{f(x)} = (\frac{\partial g_i}{\partial x_j}|_{f(x)})$ is continuous.

Since the sums and products of continuous functions are continuous, each entry of $D(h)_a = D(G)_{f(a)} D(f)_a$ is continuous.