$\mathbf{R^n}$ a vector space over $\mathbf{R}$ (or $\mathbf{C}$) with canonical basis $\{\mathbf{e_1}, ..., \mathbf{e_n}\}$ where $\mathbf{e_i} = (0, ., 0, 1, 0, ..., 0)$

Inner product on $\mathbf{R^n}$: $(\mathbf{x}, \mathbf{y}) = \mathbf{\Sigma_{i=1}^n x_i y_i}$

The basis is orthonormal: $(\mathbf{e_i}, \mathbf{e_j},) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = (\mathbf{x}, \mathbf{y})^{\frac{1}{2}}$

The *norm* of $\mathbf{x} = ||\mathbf{x}|| = \mathbf{d}(\mathbf{x}, \mathbf{0})$

$B_\epsilon^n(\mathbf{x}) = \{\mathbf{y} \in \mathbf{R^n} \mid \mathbf{d}(\mathbf{x}, \mathbf{y}) < \epsilon\}$ = ball of radius ϵ centered at $\mathbf{x}$.

$C_\epsilon^n(\mathbf{x}) = \{\mathbf{y} \in \mathbf{R^n} \mid |\mathbf{x_i} - \mathbf{y_i}| < \epsilon, \mathbf{i} = \mathbf{1}, ..., \mathbf{n}\}$ = cube of side 2ϵ centered at $\mathbf{x}$.

$\mathbf{R^1} = \mathbf{R}$, $\mathbf{R^0} = \{\mathbf{0}\}$.

I.2

$\mathbf{R^n} = \mathbf{E^n}$ where a coordinate system is defined on $\mathbf{E^n}$

A property is *Euclidean* if is does not depend on the choice of an orthonormal coordinate system.

I.3 Topological Manifolds

Defn: M is *locally Euclidean of dimension* n if for all $p \in M$, there exists an open set U_p such that $p \in U_p$ and there exists a homeomorphism $f_p : U_p \to V_p$ where $V_p \subset \mathbf{R^n}$.

Defn 3.1: An *n-manifold*, M, is a topological space with the following properties:

1.) M is locally Euclidean of dimension n.

2.) M is Hausdorff.

3.) M has a countable basis.

Give an example of a locally Euclidean space which is not Hausdorff:

Ex 3.2: If U is an open subset of an $n-$manifold, then U is also an $n-$manifold.

Ex 3.3: $S^n = \{\mathbf{x} \in \mathbf{R^{n+1}} \mid ||\mathbf{x}|| = \mathbf{1}\}$ is an _______manifold

Proof. stereographic projection:

projection:

Remark 3.5. For a "smooth" manifold, $M \subset \mathbf{R^n}$, can choose a projection by using the fact that for all $p \in M$ there exists a unit normal vector N_p and tangent plane $T_p(M)$ which varies continuously with p.

Example: smooth and non-smooth curve.

Example 3.4: The product of two manifolds is also a manifold.

Example: Torus $= S^1 \times S^1$.

Theorem 3.6: A manifold is

1.) locally connected, 2.) locally compact, 3.) a union of a countable collection of compact subsets, 4.) normal, and 5.) metrizable.

Defn: X is **locally connected at** x if for every neighborhood U of x, there exists connected open set V such that $x \in V \subset U$. X is **locally connected** if x is locally connected at each of its points.

Defn: X is **locally compact at** x is there exists a compact set $C \subset X$ and a set V open in X such that $x \in V \subset C$. X is **locally compact** if it is locally compact at each of its points.

Defn: X is **regular** if one-point sets are closed in X and if for all closed sets B and for all points $x \notin B$, there exist disjoint open sets, U, V, such that $x \in U$ and $B \subset V$.

Defn: X is **normal** if one-point sets are closed in X and if for all pairs of disjoint closed sets A, B, there exist disjoint open sets, U, V, such that $A \subset U$ and $B \subset V$.

Brouwer's Theorem on Invariance of Domain (1911). If $\mathbf{R^n} = \mathbf{R^m}$, then $n = m$.

Recall: M is *locally Euclidean of dimension* n if for all $p \in M$, there exists an open set U_p such that $p \in U_p$ and there exists a homeomorphism $f : U_p \to V_p$ where $V_p \subset \mathbf{R^n}$.

(U_p, f) is a *coordinate nbhd* of p.

Given (U_p, f) Let $q \in U \subset M$. $f(q) = (f_1(q), f_2(q), ..., f_n(q)) \in \mathbf{R^n}$ are the *coordinates* of q.

I.4 Manifolds with boundary and Cutting and Pasting

If $dim M = 0$, then $M =$

If M is connected and $dim M = 1$, then

Thm 4.1: Every compact, connected, orientable 2-manifold is homeomorphic to a sphere, or to a connected sum of tori, or to a connected sum of projective planes

Let upper half-space, $H^n = \{(x_1, x_2, ..., x_n) \in \mathbf{R^n} \mid \mathbf{x_n} \geq \mathbf{0}\}$,

$\partial H^n = \{(x_1, x_2, ..., x_{n-1}, 0) \in \mathbf{R^n}\} \sim \mathbf{R^{n-1}}$

M is a manifold with boundary if it is Hausdorff, has a countable basis, and if for all $p \in U$, there exists an open set U_p such that $p \in U_p$ and there exists a homeomorphism $f : U_p \to V_p$ one of the following holds:

i.) $V_p \subset H^n - \partial H^n$ (p is an interior point) or

ii.) $V_p \subset H^n$ and $f(p) \in \partial H^n$ (p is a boundary point).

$\partial M =$ set of all boundary points of M is an (n-1)-dimensional manifold.