Lemma: If $\mathcal{S}$ is a subbasis for a topology on X, then $\mathcal{B} = \{\cap_{i=1}^{n} S_i \mid S_i \in \mathcal{S}\}$ is a basis for a topology.

(1) Show for each $x \in X$, there is at least one basis element B containing x.

(2) Show that if $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that

$$x \in B_3 \subset B_1 \cap B_2.$$

Proof of (2):
Suppose that $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$

Find $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Let $B_3 = B_1 \cap B_2$.

Note $x \in B_3 = B_1 \cap B_2 \subset B_1 \cap B_2$.

$B_1, B_2 \in \mathcal{B}$ implies that B_i is a finite intersection of subbasis elements for $i = 1, 2$. Thus since $B_1 \cap B_2$ is the intersection of two sets each of which is a finite intersection of subbasis elements, $B_1 \cap B_2$ is a finite intersection of subbasis elements. Thus $B_3 = B_1 \cap B_2 \in \mathcal{B}$.

OR

$B_1, B_2 \in \mathcal{B}$ implies that $B_1 = \cap_{i=1}^n S_i$ and $B_2 = \cap_{i=1}^m U_i$ where $S_i, U_i \in \mathcal{S}$ for all i. Thus, $B_3 = B_1 \cap B_2 = (\cap_{i=1}^n S_i) \cap (\cap_{i=1}^m U_i) = \cap_{i=1}^{m+n} V_i$

$$\text{where } V_i = \begin{cases} S_i, & i = 1, ..., n \\ U_i & i = n+1, ..., m+n \end{cases} \text{ for all } i.$$

Thus $V_i \in \mathcal{S}$ and hence $B_3 = B_1 \cap B_2 \in \mathcal{B}$.

Defn: The **topology generated by the subbasis** $\mathcal{S}$ is the topology generated by the basis $\mathcal{B} = \{\cap_{i=1}^n S_i \mid S_i \in \mathcal{S}\}$.

Corollary: The topology generated by the subbasis $\mathcal{S}$ is the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.