Thm: If X is Hausdorff, then the diagonal, D, is closed in $X \times X$.

Need to prove D is closed in $X \times X$.

Question: How do you prove a set is closed?

a.) Show $X \times X - D$ is open.

b.) Show $D' \subset D$.

c.) Show $D = \overline{D}$.

d.) Show D is a finite union of closed sets or an arbitrary intersection of closed sets.

e.) etc.

Question: What is given?

X is Hausdorff. Hence

a.) For all $u, v \in X$ such that $u \neq v$, there exists sets U and V which are open in X such that $u \in U$, $v \in V$, $U \cap V = \emptyset$.

b.) Every finite point set in X is closed.

c.) 17.9 involving limit points, 17.10 involving sequences, etc.

Question: How can we relate what is given to what we need to prove. Which definitions/theorems involving what we need to prove are most related to which definitions/theorems of what is given.

The definition of Hausdorff gives open sets (in X). Hence we will try focusing on the definition D closed in $X \times X$ iff $X \times X - D$ is open in $X \times X$.

Concern: X Hausdorff gives open sets in X, but we need open set in $X \times X$.

We will ignore this concern for either of the following two reasons:

1.) We have to try something. Might as well give this a try. Working blindly with little motivation as to why this might work may be disconcerting, BUT it often results in the correct answer. If you don't know what to do, try something (hopefully relating what is given to what you need to prove even if weakly related). When successful, look back over your proof and see if you can figure out why it worked and how you could have come up with it with more motivation.

2.) $X \times X$ has the product topology, thus we can create the needed open set in $X \times X$ by taking the product of two open sets in X. Since X Hausdorff gives us two open sets in X, this is excellent motivation.

Thus we will try to show $X \times X - D$ is open in $X \times X$. There are many ways to show a set is open. We'll try the definition.

Take $(x, y) \in X \times X - D$ (note since it often helps to be specific, I didn't take a point p in $X \times X - D$, but instead took an ordered pair (x, y)).

We need to find a set W open in $X \times X$ such that $(x, y) \in W \subset X \times X - D$. Since basis elements are open, we could look for a basis element $U \times V$ where U and V are open in X such that $(x, y) \in U \times V \subset X \times X - D$ (note usually it is easier to find a basis element because you know what basis elements look like, but keep in mind that not all open sets are basis elements).

Looking for a basis element is particularly useful for this problem since Hausdorff gives us open sets in X and we are now looking for U, V open in X such that $x \in U$, $y \in V$, and $U \times V \subset X \times X - D$.

We will now try to apply the hypothesis that X is Hausdorff. To apply Hausdorff, we need two distinct points in X, preferably ones related to what we are trying to prove. Since $(x, y) \in X \times X - D$, x, y are points in X. Are they distinct? Since $(x, y) \notin D = \{(a, a) \mid a \in X\}$, $x \neq y$. Hence x, y are two distinct points in X. Hence there exists U, V open in X such that $x \in U$, $y \in V$, $U \cap V = \emptyset$.

Thus $(x, y) \in U \times V$. Is $U \times V \subset X \times X - D$. Suppose $(u, v) \in U \times V$. We have not yet used $U \cap V = \emptyset$. $u \in U$, $v \in V$, $U \cap V = \emptyset$ implies $u \neq v$. Hence $(u, v) \notin D$. Thus $(u, v) \in X \times X - D$.

Now you know how to relate X Hausdorff to D is closed in $X \times X$ to prove X Hausdorff implies D is closed in $X \times X$. Perhaps this relationship can also be used to prove D closed in $X \times X$ implies X Hausdorff. Sometimes a different relationship is needed to prove the other direction.

Note this proof was actually an "easy" proof. All it used was definitions. We used the definition of closed to start the proof and give us the two distinct points $x, y \in X$ so we could apply Hausdorff. We used the definition of Hausdorff to find the $U \times V$ that we needed.

Thus the proof for this direction is:

Take $(x, y) \in X \times X - D$. Since $(x, y) \notin D$, $x \neq y$. Thus there exists U, V open in X such that $x \in U$, $y \in V$, $U \cap V = \emptyset$. Thus $(x, y) \subset U \times V$. If $(u, v) \in U \times V$, then $u \in U$, $v \in V$. Thus, $U \cap V = \emptyset$ implies $u \neq v$. Hence $(u, v) \notin D$. Thus $(u, v) \in X \times X - D$. Thus, $U \times V \subset X \times X - D$. Hence $X \times X - D$ is open in $X \times X$. Thus D is closed in $X \times X$.