18. Continuous Functions

Defn: $f : X \to Y$ is an imbedding of X in Y iff $f : X \to f(X)$ is a homeomorphism.

Thm 18.2

(a.) (Constant function) The constant map $f : X \to Y$, $f(x) = y_0$ is continuous.

(b.) (Inclusion) If A is a subspace of X, then the inclusion map $f : A \to X$, $f(a) = a$ is continuous.

(c.) (Composition) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

(d.) (Restricting the Domain) If $f : X \to Y$ is continuous and if A is a subspace of X, then the restricted function $f|_A : A \to Y$, $f|_A(a) = f(a)$ is continuous.

(f.) (Local formulation of continuity) If $f : X \to Y$ and $X = \cup U_\alpha$, U_α open where $f|_{U_\alpha} : U_\alpha \to Y$ is continuous, then $f : X \to Y$ is continuous.

(g) (The pasting lemma) If $f : X \to Y$ and $X = \cup_{i=1}^n A_i$, A_i closed where $f|_{A_i} : A_i \to Y$ is continuous, then $f : X \to Y$ is continuous.

Thm 18.3 (The pasting lemma): Let $X = A \cup B$ where A, B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cap B$, then $h : X \to Y$,

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases} \text{ is continuous.}$$

Thm 18.4: Let $f : A \to X \times Y$ be given by the equations $f(a) = (f_1(a), f_2(a))$ where $f_1 : A \to X$, $f_2 : A \to Y$. Then f is continuous if and only if f_1 and f_2 are continuous.

Note the above also holds for arbitrary products in the product topology.

Thm 36.1 (Existence of partitions of unity). Let $\{U_1, ..., U_n\}$ be a finite indexed open cover of X and let X be T_4. Then there exists a partition of unity dominated by $\{U_i\}$

Thm 36.2 If X is a compact m-manifold, then X can be imbedded in R^N for some positive integer N.