Definition: A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X.

Lemma: If $\mathcal{S}$ is a subbasis for a topology on X, then $\mathcal{B} = \{\cap_{i=1}^{n} S_i \mid S_i \in \mathcal{S}\}$ is a basis for a topology.

Note that $\mathcal{B}$ is a collection of subsets of X ($S_i \subset X$ implies $\cap_{i=1}^{n} S_i \subset X$).

(1) Show for each $x \in X$, there is at least one basis element B containing x.

Proof: Take $x \in X$. Find $B \in \mathcal{B}$ such that $x \in B$.

We will first show that $\mathcal{S} \subset \mathcal{B}$: Suppose that $S_1 \in \mathcal{S}$. Then $S_1 = \cap_{n=1}^{1} S_1$. Thus S_1 is a finite intersection of elements of $\mathcal{S}$ since 1 is a finite number. Hence $S_1 \in \mathcal{B}$.

Since $\mathcal{S}$ is a collection of subsets of X whose union equals X, there exists an $S \in \mathcal{S} \subset \mathcal{B}$ such that $x \in S$. Hence, there is at least one basis element B containing x.

(2) Show that if $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that

$$x \in B_3 \subset B_1 \cap B_2.$$

The topology $\mathcal{T}$ generated by a basis $\mathcal{B}$ is defined as follows: U is open if and only if for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

Negation:
U is NOT open in $\mathcal{T}$ iff the following is false: for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

U is NOT open in $\mathcal{T}$ iff there exists an $x \in U$, such that the following is false: there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

U is NOT open in $\mathcal{T}$ iff there exists an $x \in U$, such that for all $B \in \mathcal{B}$ the following is false: $x \in B \subset U$.

U is NOT open in $\mathcal{T}$ iff there exists an $x \in U$, such that for all $B \in \mathcal{B}$ the following is false: $x \in B$ and $B \subset U$.

U is NOT open in $\mathcal{T}$ iff there exists an $x \in U$, such that for all $B \in \mathcal{B}$ either $x \notin B$ OR $B \not\subset U$.

U is NOT open in $\mathcal{T}$ iff there exists an $x \in U$, such that for all $B \in \mathcal{B}$, $x \in B$ implies $B \not\subset U$.