p91: 4, 6

4.) Let W be open in $X \times Y$.

Show $\pi_1(W)$ is open in X.

Let $x_0 \in \pi_1(W)$. Find U open in X such that $x_0 \in U \subset \pi_1(W)$.

$x_0 \in \pi_1(W)$ implies there exists an $(x_0, y_0) \in W$ such that $\pi_1(x_0, y_0) = x_0$. W open implies there exists a basis element $U \times V$ where U is open in X and V is open in Y and $(x_0, y_0) \in U \times V \subset W$ Thus $\pi_1((x_0, y_0)) \in \pi_1(U \times V) \subset \pi_1(W)$. Hence $x_0 \in U \subset \pi_1(W)$.

6.) R^2 has the product topology. Thus by thm 15.1, $\{B_1 \times B_2 \mid B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$ is basis for R^2 where $\mathcal{B}_i$ are bases for R. By problem #8, p. 83, $\mathcal{B} = \{(a, b) \mid a < b,\ a \text{ and } b \text{ rational }\}$ is a basis for R. Thus $\{(a, b) \times (c, d) \mid a < b \text{ and } c < d,\ a, b, c, d \text{ rational }\}$ is a basis for R^2.

p91: 1, 3, 8

1.) Let $\mathcal{T}_X$ be the topology on X.

Let $\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}_X\}$

Let $\mathcal{T}_A = \{A \cap U \mid U \in \mathcal{T}_X\}$

Let $\mathcal{T}'_A = \{A \cap U \mid U \in \mathcal{T}_Y\}$

Show $\mathcal{T}_A = \mathcal{T}'_A$

Let $W \in \mathcal{T}_A$. Then there exists a $U \in \mathcal{T}_X$ such that $W = A \cap U$. $U \in \mathcal{T}_X$ implies $Y \cap U \in \mathcal{T}_Y$. Since $A \subset Y$, $A \cap (Y \cap U) = (A \cap Y) \cap U = A \cap U = W$. Thus $W \in \mathcal{T}'_A$.

Hence $\mathcal{T}_A \subset \mathcal{T}'_A$

Let $W \in \mathcal{T}'_A$. Then there exists a $U \in \mathcal{T}_Y$ such that $W = A \cap U$. $U \in \mathcal{T}_Y$ implies there exists a $V \in \mathcal{T}_X$ such that $U = Y \cap V$. Since $A \subset Y$, $W = A \cap U = A \cap Y \cap V = A \cap V$. Thus $W \in \mathcal{T}_A$.

Hence $\mathcal{T}'_A \subset \mathcal{T}_A$

Therefore $\mathcal{T}_A = \mathcal{T}'_A$

3.) A and E are open in R. B, C, D are not open in R. A, B, E are open in $[-1, 1]$. C, D are not open in $[-1, 1]$.

$A = (-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1)$ is open in R since A is the union of two basis element.

$B = [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$. Show B is not open in R. Show that there exists an $x \in B$ such that for all basis elements, B_x such that $x \in B_x$, $B_x \not\subset B$.

Let $x = 1$. Suppose $1 \in (a, b)$. Thus $a < 1 < b$. Hence $a < 1 < \frac{1+b}{2} < b$. Thus $\frac{1+b}{2} \in (a, b)$, but $\frac{1+b}{2} \notin [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$. Thus, $(a, b) \not\subset B$. Thus B is not open.

Similarly C (take $x = \frac{1}{2}$) and D (take $x = \frac{1}{2}$ or 1) are not open.

Take $x \in E = \{x \mid 0 < |x| < 1 \text{ and } \frac{1}{x} \notin Z_+\}$. If $x < 0$, then $x \in (-1, 0) \subset E$. If $x \not< 0$. Then $x \neq \frac{1}{n}$ for any $n \in Z_+$ and $x \neq 0$ since $x \in E$. Let $F = \{n \in Z_+ \mid \frac{1}{n} < x\}$. Since F is bounded below by 0, $inf\ F$ exists in Z since Z has the greatest lower bound property. Let $N = inf\ F$. Then $\frac{1}{N} < x$. $N - 1 < N$ implies that $N - 1 \notin F$. Since $x < 1$, $N \neq 1$, and thus $N - 1 \in Z_+$. Hence $\frac{1}{N-1} \geq x$ and since $x \neq \frac{1}{N-1}$, $\frac{1}{N-1} > x$. Thus, $x \in (\frac{1}{N}, \frac{1}{N-1}) \subset E$. Thus E is open in R.

Alternate proof that E is open in R. $E = (-1, 0) \cup [\cup_{n=1}^{\infty} (\frac{1}{n+1}, \frac{1}{n})]$. Thus E is open since it is the union of open sets.

Question: Can you prove that $E = (-1, 0) \cup [\cup_{n=1}^{\infty} (\frac{1}{n+1}, \frac{1}{n})]$.

A open in R and $A = A \cap [-1, 1]$ implies A is open in $[-1, 1]$.

$(-3, -\frac{1}{2}) \cup (\frac{1}{2}, 5)$ open in R and $B = [(-3, -\frac{1}{2}) \cup (\frac{1}{2}, 5)] \cap [-1.1]$ implies B is open in $[-1.1]$.

$C = (-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1)$ Show C is not open. Show that there exists an $x \in C$ such that for all basis elements, B_x such that $x \in B_x$, $B_x \not\subset C$.

Let $x = \frac{1}{2}$. Suppose $\frac{1}{2} \in (a, b) \cap [-1, 1]$. Let $a' = max\{a, -\frac{1}{2}\}$. Thus $a \leq a' < \frac{1}{2} < b$. Hence $a < \frac{\frac{1}{2}+a'}{2} < \frac{1}{2} < b$.

Also $-\frac{1}{2} \leq a' < \frac{1}{2}$ implies $-1 < -\frac{1}{2} < \frac{\frac{1}{2}+a'}{2} < \frac{1}{2} < 1$ Thus $\frac{\frac{1}{2}+a'}{2} \in (a, b) \cap [-1, 1]$, but $\frac{\frac{1}{2}+a'}{2} \notin (-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1)$. Thus, $(a, b) \cap [-1, 1] \not\subset C$. Thus C is not open.

Similarly D is not open.

E open in R and $E = E \cap [-1, 1]$ implies E is open in $[-1, 1]$

8.) Let R^2 have the topology generated by $\mathcal{T}_X \times \mathcal{T}_Y$.

Let L be a line with slope 0. Then $L = R \times \{y_0\}$ for some $y_0 \in R$. Note that $L = R \times \{y_0\}$ has the subspace topology by hypothesis. By thm 16.3, the subspace topology on $L = R \times \{y_0\}$ as a subspace of R^2 is the same as the product topology on $R \times \{y_0\}$ where R is a subspace of $(R, \mathcal{T}_X)$ and $\{y_0\}$ is a subspace of $(R, \mathcal{T}_Y)$. Thus, $R \times \{y_0\}$ has the topology generated by $\{U \times V \mid U \in \mathcal{T}_X, V \in \{\emptyset, \{y_0\}\}\}$. Since $A \times \emptyset = \emptyset$, $\{U \times V \mid U \in \mathcal{T}_X, V \in \{\emptyset, \{y_0\}\}\} = \{U \times V \mid U \in \mathcal{T}_X, V = \{y_0\}\}$. Thus the topology on L is equivalent to $\mathcal{T}_X$. Thus L inherits the lower limit topology as a subspace of $R_l \times R$ and L inherits the lower limit topology as a subspace of $R_l \times R_l$.

Similarly if L is a line with vertical slope, the the topology on $L = \mathcal{T}_Y$. Thus L inherits the standard topology as a subspace of $R_l \times R$ and L inherits the lower limit topology as a subspace of $R_l \times R_l$.

Let R^2 have the topology generated by $R \times R_l$.

Let L be a line with positive slope. Let $f(x) = mx + b$ where $m > 0$. $L = \{(x, f(x)) \mid x \in R\}$. $(a, b) \times [c, d)$ is a basis element for the topology generated by $R \times R_l$.

If $f(a) \geq d$ or $f(b) \leq c$, then

$\{(x, f(x)) \mid x \in R\} \cap (a, b) \times [c, d) = \emptyset$

If $f(a) < c$ and $f(b) \geq d$ then

$\{(x, f(x)) \mid x \in R\} \cap (a, b) \times [c, d) = [(f^{-1}(c), c), (f^{-1}(d), d))$ where $[(f^{-1}(c), c), (f^{-1}(d), d)) = \{(x, f(x)) \mid f^{-1}(c) \leq x < f^{-1}(d)\}$

If $f(a) < c$, $c < f(b) \leq d$ then

$\{(x, f(x)) \mid x \in R\} \cap (a, b) \times [c, d) = [(f^{-1}(c), c), (b, f(b)))$ where $[(f^{-1}(c), c), (b, f(b))) = \{(x, f(x)) \mid f^{-1}(c) \leq x < b, \}$

Thus since f is onto, every half open interval in L is open. Thus the topology on L is equal to or finer that the lower limit topology.

If $c \leq f(a) < d$ and $f(b) \geq d$ then

$\{(x, f(x)) \mid x \in R\} \cap (a, b) \times [c, d) = ((a, f(a)), (f^{-1}(d), d))$ where $((a, f(a)), (f^{-1}(d), d)) = \{(x, f(x)) \mid a < x < f^{-1}(d)\}$

If $c \leq f(a) < d$, $c < f(b) \leq d$ then

$\{(x, f(x)) \mid x \in R\} \cap (a, b) \times [c, d) = ((a, f(a)), (b, f(b))$ where $((a, f(a)), (b, f(b)) = \{(x, f(x)) \mid a < x < b, \}$

Thus, every open interval in L is open. Note that open intervals are open in the lower limit topology.

Since we have determined for every possible intersection between L and basis elements of $R \times R_l$, the collection of half open intervals is a basis for the topology on L. Thus L inherits the lower limit topology as a subspace of $R \times R_l$

Similarly, L inherits the lower limit topology as a subspace of $R_l \times R_l$

Similarly, if L is a line with negative slope, L inherits the upper limit topology as a subspace of $R \times R_l$

Let R^2 have the topology generated by $R_l \times R_l$

Let L be a line with negative slope. $L = \{(x, y) \mid y = mx + b\}$ where $m < 0$. Let $f(x) = mx + b$. $[x, x + 1) \times [f(x), f(x) + 1) \cap L = \{(x, f(x))\}$.

Thus every point in L is open. Thus the topology on L is discrete topology.