33.2a) Suppose $p^{-1}(U)$ is open in X. Then $f^{-1}(p^{-1}(U))$ is open in Y. Since $f \circ p = id$, $f^{-1}(p^{-1}(U)) = U$. Thus U is open in Y.

Hence U is open in Y if and only if $p^{-1}(U)$ is open in X. Thus p is a quotient map IF p is surjective.

Note $p : R \to R^2$, $p(x) = (x, 0)$ is continuous. Note $\pi_1 : R^2 \to R$ is continuous and $\pi_1(p(x)) = \pi_1(x, 0) = x$.

Thus p satisfies the hypothesis of this problem, but is NOT a quotient map since it is not surjective.

b.) Define $f : A \to X$ by $f(a) = a$. Since f is obtained from the identity map $i : A \to A$ by expanding the co-domain, f is continuous. $r(f(a)) = r(a) = a$ for all $a \in A$. Hence $r \circ f$ is the identity map of A. Since r is continuous and onto, r is a quotient map by part a.

43.8.) $e : X \times C(X, Y) \to Y$, $e(x, f) = f(x)$. If d is a metric for Y and $C(X, Y)$ has the uniform topology, then e is continuous.

Since d and $\overline{d}$ generate the same topology on Y and continuity is a topological property, we will use $\overline{d}$ to prove the continuity of e. Hence, we will use the following notation: $B_Y(y, \epsilon) = \{z \in Y \mid \overline{d}(y, z) < \epsilon\}$ and $B_{\overline{\rho}}(h, \epsilon) = \{g \in C(C, Y) \mid \overline{\rho}(h, g) < \epsilon\}$

Take U open in Y and $(x, f) \in e^{-1}(U)$. Then $e(x, f) = f(x) \in U$. Hence there exists an $r > 0$ such that $B_Y(f(x), r) \subset U$.

$f \in C(X, Y)$ implies $f^{-1}(B_Y(f(x), \frac{r}{2}))$ is open in X

Thus $f^{-1}(B_Y(f(x), \frac{r}{2}) \times B_{\overline{\rho}}(f, \frac{r}{2})$ is open in $X \times C(X, Y)$.

Suppose $(z, g) \in f^{-1}(B_Y(f(x), \frac{r}{2})) \times B_{\overline{\rho}}(f, \frac{r}{2})$.

$z \in f^{-1}(B_Y(f(x), \frac{r}{2}))$ implies $f(z) \in B_Y(f(x), \frac{r}{2})$. Thus $d(f(z), f(x)) < \frac{r}{2}$

$g \in B_{\overline{\rho}}(f, \frac{r}{2})$ implies $\overline{\rho}(f, g) < \frac{r}{2}$. Hence $d(g(z), f(z)) < \frac{r}{2}$

$d(g(z), f(x)) \leq d(g(z), f(z)) + d(f(z), f(x)) < \frac{r}{2} + \frac{r}{2} = r$. Thus $e(z, g) = g(z) \in B_Y(f(x), r$.

Thus $(z, g) \in e^{-1}(B_Y(f(x), r)) \subset e^{-1}(U)$.

Hence e is continuous.

43.10) If $h : X \to Y$ $h' : X \to Y'$ are isometric imbedding of metric space (X, d) into complete metric spaces (Y, D) and (Y', D'), respectively then there is an isometry of $(\overline{h(X)}, D)$ with $(\overline{h'(X)}, D')$ that equals $h'h^{-1}$ on $h(X)$.

Proof: $h'h^{-1} : h(X) \to h'(X)$

Claim: If $f : A \to B$ is continuous, and $A \subset X$ where X is first countable and Y is metrizable, then f can be extended to $F : \overline{A} \to \overline{B}$ such that $F(a) = f(a)$ for all $a \in A$.

Proof: Suppose $f : A \to B$ is continuous. Then if $x \in \overline{A}$, there exists $x_n \in A$ such that $x_n \to x$.

Define $F : \overline{A} \to \overline{B}$ by $F(x) = limf(x_n)$ where x_n is any sequence in A such that $x_n \to x$.

Claim: F is well-defined.

Suppose that $x_n \to x$ and $w_n \to x$. Then $d(limf(x_n), limf(w_n)) = limd(f(x_n), f(w_n)) = 0$ (Can you prove this last equality?). Hence $limf(x_n) = limf(w_n)$.

Since the constant sequence $a, a, a, ...$ converges to a, $F(a) = limf(a) = f(a)$. Hence F extends f.

Hence $h'h^{-1} : h(X) \to h'(X)$ can be extended to a function $F : (\overline{h(X)}, D) \to (\overline{h'(X)}, D')$ where $F(y) = lim[h'h^{-1}(y_n)]$ where $y_n \in h(X)$ and $y_n \to y$.

Claim F is an isometry:

Let $y_n \to z$, $y_n \in h(X)$ and $z_n \to z$, $z_n \in h(X)$

Then $D'(F(y), F(z)) = D'(limh'h^{-1}(y_n), limh'h^{-1}(z_n)) = limD'(h'h^{-1}(y_n), h'h^{-1}(z_n)) = lim(D(h^{-1}(y_n), h^{-1}(z_n))) = limD'(y_n, z_n) = D'(y, z)$ since D' is continuous and h' and h^{-1} are isometries. ∎