Homework 5 [adapted from latex HW of Colin McKinney]

5.6A Show that if $A \subset B$, then $\overline{A} \subset \overline{B}$.

Take $x \in \overline{A}$. By Theorem 17.5, every neighborhood U of x intersects A. Since $A \subset B$, U also intersects B, which by Theorem 17.5 implies that $x \in \overline{B}$.

5.6B Show $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

$A \subset \overline{A}$ and $B \subset \overline{B}$. Hence, $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$. Since $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$, $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$ by 6c.

5.6C Show $\cup\overline{A_\alpha} \subset \overline{\cup A_\alpha}$; give an example where equality fails.

Fix $x \in \cup\overline{A_\alpha}$. This implies that $x \in \overline{A_\alpha}$ for some α. By Theorem 17.5, this implies that every neighborhood U of x intersects A_α for some α. Since $A_\alpha \subset \cup A_\alpha$, we know that every neighborhood U of x intersects $\cup A_\alpha$. By Theorem 17.5, then, $x \in \overline{\cup A_\alpha}$. Hence $\cup\overline{A_\alpha} \subset \overline{\cup A_\alpha}$.

Specific counter-example where equality fails:

8 $A \cap B \subset A$. Hence $\overline{A \cap B} \subset \overline{A}$. Similarly $\overline{A \cap B} \subset \overline{B}$. Hence $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$

Alternatively, $\overline{A \cap B}$ is the smallest closed set containing $A \cap B$. $A \subset \overline{A}$ and $B \subset \overline{B}$. Hence $A \cap B \subset \overline{A} \cap \overline{B}$. Thus, $\overline{A} \cap \overline{B}$ is a closed set containing $A \cap B$. Hence $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

Specific counter-example where equality fails:

$\cap A_\alpha \subset A_{\alpha_0}$. Hence $\overline{\cap A_\alpha} \subset \overline{A_{\alpha_0}}$ for all α_0. Hence $\overline{\cap A_\alpha} \subset \cap\overline{A_\alpha}$

Specific counter-example where equality fails:

$\overline{A} - \overline{B} = \overline{A} \cap (\overline{B})^c \subset \overline{A} \cap B^c$ since $B \subset \overline{B}$ implies $(\overline{B})^c \subset B^c$. $\overline{A} \cap B^c \subset \overline{A} \cap \overline{(B^c)} = \overline{A \cap B^c} = \overline{A - B}$. Hence $\overline{A} - \overline{B} \subset \overline{A - B}$.

Specific counter-example where equality fails:

5.11 Show that the product of two Hausdorff spaces is Hausdorff.

Let A and B be two Hausdorff spaces. Take $x_1 \times y_1$, $x_2 \times y_2$ in $A \times B$ such that $x_1 \times y_1 \neq x_2 \times y_2$. Then either $x_1 \neq x_2$ or $y_1 \neq y_2$. Without loss of generality assume $x_1 \neq x_2$. Since A is Hausdorff, there exists U_1, U_2 such that U_1 is a neighborhood of x_1 in A and U_2 is a neighborhood of x_2 in A, where U_1 and U_2 are disjoint. Then $U_1 \times B$ will be a neighborhood of $x_1 \times y_1$ in $A \times B$, and $U_2 \times B$ will be a neighborhood of $x_2 \times y_2$ in $A \times B$. Since U_1 and U_2 are disjoint, it follows that $U_1 \times B$ and $U_2 \times B$ are disjoint. Since we have found disjoint neighborhoods for two arbitrary points in $A \times B$, by the definition of Hausdorff, $A \times B$ is Hausdorff.

5.12 Show that a subspace of two Hausdorff spaces is Hausdorff.

Let X be a Hausdorff space, and let Y be a subspace of X. Let x_1 and x_2 be elements of Y such that $x_1 \neq x_2$. Since X is Hausdorff, there exist disjoint neighborhoods U_1 and U_2 in X of x_1 and x_2, respectively. Hence a set containing x_1 in Y is $V_1 = U_1 \cap Y$, which is open in Y by definition of the subspace topology on Y. Thus V_1 is a neighborhood of x_1 in Y. Similarly, a set containing x_2 in Y is $V_2 = U_2 \cap Y$, which is open in Y by the definition of the subspace topology on Y. Thus V_2 is a neighborhood of x_2 in Y. Now since $V_1 \subset U_1$ and $V_2 \subset U_2$, and U_1 and U_2 are disjoint, it follows that V_1 and V_2 are disjoint. Thus, Y is Hausdorff.