Topology HW answers (many answers adapted from latexed HW of Colin B. McKinney)

p. 111:

2.) Suppose that $f : X \mapsto Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

It is NOT necessarily true that $f(x)$ is a limit point of $f(A)$ if x is a limit point of the subset A of X. The constant function $f : R \to R$, $f(x) = 0$ is continuous. 5 is a limit point of R, but $f(5) = 0$ is not a limit point of $f(R) = \{0\}$ since $\{0\}$ has no limit points.

9.) Let A_α be a finite collection of subsets of X; let $X = \cup_\alpha A_\alpha$. Let $f : X \mapsto Y$; suppose that $f|A_\alpha$ is continuous for each α.

Since each $f|A_\alpha$ is continuous, if we choose any set V closed in Y, $(f|A_\alpha)^{-1}(V) = f^{-1}(V) \cap A_\alpha$ is closed in A_α. Since A_α in closed in X, $f^{-1}(V) \cap A_\alpha$ is closed in X. Hence $\bigcup_\alpha (f^{-1}(V) \cap A_\alpha) = f^{-1}(V) \cap (\bigcup_\alpha A_\alpha) = f^{-1}(V) \cap X = f^{-1}(V)$ is closed since finite unions of closed sets are closed. Hence f is continuous.

p. 118

3.) Suppose X_α is Hausdorff for all α. Show ΠX_α is Hausdorff

Take $(x_\alpha)_{\alpha \in J}$ and $(y_\alpha)_{\alpha \in J} \in \Pi X_\alpha$ such that $(x_\alpha)_{\alpha \in J} \neq (y_\alpha)_{\alpha \in J}$. Then there exists α_0 such that $x_{\alpha_0} \neq y_{\alpha_0}$. Since X_{α_0} is Hausdorff there exists U, V open in X_{α_0} such that $U \cap V = \emptyset$ and $x_{\alpha_0} \in U$, $y_{\alpha_0} \in V$. Then $(x_\alpha)_{\alpha \in J} \in \Pi U_\alpha$ and $(y_\alpha)_{\alpha \in J} \in \Pi V_\alpha$ are disjoint open sets in the box and product topology where $U_\alpha = \begin{cases} U & \alpha = \alpha_0 \\ X & \alpha \neq \alpha_0 \end{cases}$.

5: Theorem: Let $f : A \mapsto \Pi_{a \in J} X_\alpha$ be given by the equation $f(a) = (f_\alpha(a))_{\alpha \in J}$, where $f_\alpha : A \mapsto X_\alpha$ for each α. Let ΠX_α have the product topology. Then the function f is continuous if and only if each function f_α is continuous. One of the implications stated in the preceeding theorem holds for the box topology. Which one?

The following implication holds: if f is continuous, then each f_α is continuous. This is true since each f_α is a function that restricts the range of f. The reverse implication, however, is not necessarily true, as shown in example 2, pg. 117.

Prove Theorem 20.4: The uniform topology on R^J is finer than the product topology and coarser than the box topology; these three topologies are different if J is infinite.

8.1A Show the uniform topology is finer than the product topology.

Take a basis element in the product topology: ΠU_α, where each U_α is open in R. Show ΠU_α is open in the uniform topology. Take a point x in R^J; represent x as $x = (x_\alpha)$ for $\alpha \in J$. By definition of the product topology, there only finitely many of the $U_\alpha \neq R$, so denote let $\alpha_1, \alpha_2, ..., \alpha_n$ be the values of α for which $U_\alpha \neq R$. Next, for each i, choose $\epsilon_i > 0$ so that $B_{\overline{d}}(x_{\alpha_i}, \epsilon_i) \subset U_{\alpha_i}$, which is possible since by definition each U_{α_i} is open in R.

Claim: if we let $\epsilon = \min\{\epsilon_1, ..., \epsilon_n\}$, $x \in B_{\overline{\rho}}(x, \epsilon) \subset \Pi U_\alpha$.

If a point $z \in B_{\overline{\rho}}(x, \epsilon)$ then $\overline{d}(x_\alpha, z_\alpha) \leq \overline{\rho}(x, z) < \epsilon \leq \epsilon_i$ for all i and α. Hence $z_{\alpha_i} \in B_{\overline{\rho}}(x_{\alpha_i}, \epsilon_i) \subset U_{\alpha_i}$ for all α_i and $z_\alpha \in R = U_\alpha$ for all $\alpha \neq \alpha_1, ..., \alpha_n$, so $z \in \Pi U_\alpha$. Hence the uniform topology is finer than the product topology.

8.1B For J infinite, show that the uniform topology is different than the product topology.

$B_{\overline{\rho}}(\mathbf{0}, 1)$ is not open in the product topology when J is infinite since $B_{\overline{\rho}}(\mathbf{0}, 1) \subset \Pi_{\alpha \in J}(-1, 1)$.$\blacksquare$ Thus, the uniform topology is different than the product topology when J is infinite.

Alternatively, you can generalize problem 4 on p. 127.

8.1C Show that the box topology is finer than the uniform topology.

Note the book's proof is incorrect.

Begin by letting B be the ϵ_1-ball around y in the uniform topology. Show B is open in the box topology. Take $x \in B$. Then there exists a $B_{\overline{\rho}}(x, \epsilon) \subset B_{\overline{\rho}}(y, \epsilon_1)$ Then we could define a neighbourhood of x in the box topology as $U = \prod(x_\alpha - \frac{1}{2}\epsilon, x_\alpha + \frac{1}{2}\epsilon)$.

Claim U is contained in B. For any element z of U, $\overline{d}(x_\alpha, z_\alpha) < \frac{1}{2}\epsilon$ for all α. This implies that $\overline{\rho}(x, z) \leq \frac{1}{2}\epsilon < \epsilon$, implying that the box topology is indeed finer than the uniform topology.

8.1D Show that for J infinite, the uniform topology and the box topology are different.

Take $I \subset J$ where I is a countable subset of J. Let $x_{n_\alpha} = \begin{cases} \frac{1}{n} & \alpha \in I \\ 0 & \alpha \notin I \end{cases}$.
If $\mathbf{x_n} = \Pi_{\alpha \in J} x_{n_\alpha}$, then $\overline{\rho}(\mathbf{x_n}, \mathbf{0}) = sup\{\frac{1}{n}, 0\} = \frac{1}{n}$. Hence $\mathbf{x_n} \to \mathbf{0}$ in the uniform topology, but $\mathbf{x_n}$ does not converge to $\mathbf{0}$ in the box topology. Thus, the uniform topology and the box topology are different when J is infinite.

8.2: Using the metric D defined in Theorem 20.5, describe $B_D(0, \frac{1}{3})$.

First off we note that $D(0, x) < \frac{1}{3}$ for x to be an element of B_D. But $D(0, x) = sup\{\frac{min\{d(0, x_i), 1\}}{i}\} = \sup\{\frac{min\{|x_i|, 1\}}{i}\} < \frac{1}{3}$ for each i. For $i = 1$, the first coordinate, x_1 must satisfy $|x_1| < \frac{1}{3}$. For $i = 2$, the second coordinate, x_2 must satisfy $|x_2| < \frac{2}{3}$. For $i = 3$, the third coordinate, x_3 must satisfy $|x_3| < 1$. For $4 \leq i$, the ith coordinate can be any real number. Hence

$B_D(0, \frac{1}{3}) = (-\frac{1}{3}, \frac{1}{3}) \times (-\frac{2}{3}, \frac{2}{3}) \times (-1, 1) \times R \times R \times ...$

p. 145:

Repeat 4A for the equivalence relation $x_0 \times y_0 \ x_1 \times y_1$ if $x_0^2 + y_0^2 = x_1^2 + y_1^2$.

$f : X* \to [0, \infty)$, $f((x, y)) = \sqrt{(x^2 + y^2)}$ is a homeomorphism where $[0, \infty)$ is a subspace of R. Thus $X*$ is equivalent to the subspace $[0, \infty)$ of R.

pg 152

1.) Let $\mathcal{T}'$ and $\mathcal{T}$ be two topologies on X, where $\mathcal{T}'$ is finer than $\mathcal{T}$. What does the connectedness of X in one imply about its connectedness in the other?

R_l is finer than R which is finer than R with the indiscrete topology, R and R with the indiscrete topology are connected but R_l is not connected by problem 7.

Thus connectedness of X in $\mathcal{T}$ does not imply anything about the connectedness of X in $\mathcal{T}'$

Suppose $\mathcal{T}$ is not connected, then $X = U \cup V$ where $U, V \in \mathcal{T} \subset \mathcal{T}'$. Thus, $\mathcal{T}'$ is not connected. Hence connectedness of X in $\mathcal{T}'$ implies connectedness of X in $\mathcal{T}$.

7.) R_l is not connected since $R_l = (\infty, 0) \cup [0, \infty)$ and $(\infty, 0)$ and $[0, \infty)$ are both open in R_l.

p. 157

8A: If two spaces X and Y are path connected, is the space $X \times Y$ path connected?

Let $a \times b$ and $c \times d$ be distinct points in $X \times Y$. Since X is path connected, there exists a path $\gamma_1 : [0,1] \to X$ from a to c. Similarly, since Y is path connected, there exists a path $\gamma_2[0,1] \to Y$ from b to d, Hence $\gamma = (\gamma_1, \gamma_2) : [0,1] \to X \times Y$ is a path from $a \times b$ to $c \times d$, so $X \times Y$ is indeed path connected.

8B, If $A \subset X$ and A is path connected, is $\overline{A}$ also path connected?

Not necessarily. Take $X = R^2$ and $A = S$, where S is the topologist's sine curve given in example 7, pg 157. S is path connected, but $\overline{S}$ is not.

8CIf $f : X \to Y$ is continuous and X is path connected, is $f(X)$ necessarily path connected?

Fix x, y in $f(X)$. Take $c \in f^{-1}(x)$ and $d \in f^{-1}(y)$, the inverse images of x and y. Then since X is path connected, there exists a continuous function $\gamma_X : [a,b] \mapsto X$, where $[a,b]$ is a subset of R such that $\gamma_X(a) = c$ and $\gamma_X(b) = d$. Since both f and γ_X are continuous, the composition of f with γ_X will also be a continuous function. Let this function be γ. Thus $\gamma = f \circ \gamma_X : [a,b] \mapsto f(X)$, where $\gamma(a) = f \circ \gamma_X(a) = f(c) = x$ and $\gamma(b) = f \circ \gamma_X(b) = f(d) = y$, so $f(X)$ is path connected.

8D If $\{A_\alpha\}$ is a collection of path-connected subspaces of X, and if $\cap A_\alpha \neq \emptyset$, is $\cup A_\alpha$ necessarily path connected?

Since $\cap A_\alpha \neq \emptyset$, by the definition of intersection there exists a point y such that $y \in A_\alpha$ for all α. Let x, z be two points in $\cup A_\alpha$. x therefore must be an element of A_α for some α by the definition of union; similarly for z. Let $x \in A_1$ and $z \in A_2$. Since A_1 is path connected, and $x, y \in A_1$, there exists a path $\gamma_1 : [0,1] \to A_1$ between x and y. Similarly, there exists a path $\gamma_2 : [1,2] \to A_2$ between y and z in A_2. Hence $\gamma[0,2] \to \cup A_\alpha, \gamma(x) = \begin{cases} \gamma_1(x) & x \in [0,1] \\ \gamma_2(x) & x \in [1,2] \end{cases}$ is a path connecting x and z via y, a point in common.