p. 83: 3 [adapted from latex HW of Colin McKinney]

Show that the collection $\mathcal{T}_c$ given in Example 4 of §12 is a topology on the set $X$.

$\mathcal{T}_c := \{U | X - U \text{ is countable or all of } X\}$. First, we must ensure that $X$ and $\emptyset$ are elements of $\mathcal{T}_c$. For $U = \emptyset$, $X - \emptyset = X$. Thus, $\emptyset \in \mathcal{T}_c$. For $U = X$, $X - X = \emptyset$. The null set is finite, and so it is also countable. Hence $X \in \mathcal{T}_c$.

We must also show that $\mathcal{T}_c$ is closed under arbitrary unions. Take $U_\alpha \in \mathcal{T}_c$. Since $X - U_\alpha$ can be either countable or all of $X$, we must consider two cases.

**Case 1** : There exists an $\alpha_0$ such that $X - U_{\alpha_0}$ countable. $X - \cup U_\alpha = \cap(X - U_\alpha)$, and since $X - U_{\alpha_0}$ is countable, and $\cap(X - U_\alpha) \subset X - U_{\alpha_0}$, it follows that $\cap(X - U_\alpha)$ is countable. Hence $X - \cup U_\alpha$ is countable, and so in this case $\cup U_\alpha \in \mathcal{T}_c$.

**Case 2** : $X - U_\alpha$ is all of X for all $\alpha$. Hence $U_\alpha = \emptyset$ for all $\alpha$. It follows that $\cup U_\alpha = \emptyset \in \mathcal{T}_c$

Hence $\mathcal{T}_c$ is closed under arbitrary unions.

Lastly we must show that $\mathcal{T}_c$ is closed under finite intersections. Take $U_i \in \mathcal{T}_c$.

**Case 1** : $X - U_i$ countable for all $i$. A finite union of countable sets is also countable. By Problem 1.0 of Homework 1, we may write

$$X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i) \tag{1}$$

Since $\cup(X - U_i)$ is countable, the left hand side of (1) is also countable. Thus, $\cup_{i=1}^{n} U_i \in \mathcal{T}_c$ in this case.

**Case 2** : $X - U_{i_0}$ is all of $X$ for some $i_0 \in \{1, ..., 1\}$. Hence $U_{i_0} = \emptyset$, and $\cap_{i=1}^{n} U_i = \emptyset \in \mathcal{T}_c$.

Thus, $\mathcal{T}_c$ is closed under finite intersections. $\mathcal{T}_c$ has thus satisfied all the requirements to be a topology.

Is $\mathcal{T}_\infty := \{U | X - U \text{ is infinite, empty, or all of } X\}$ a topology on $X$?

If $X$ is infinite, $U_x = \{x\} \in \mathcal{T}_\infty$, where $x$ is any element of $X$ since $X - U_x$ would remain infinite. However, let us define an indexing set $Y := X - \{x_1, x_2\}$, where $x_1$ and $x_2$ are two distinct elements of $X$. Hence

$$X - \bigcup_{x \in Y} U_x = \{x_1, x_2\}. \tag{2}$$

Since $\{x_1, x_2\}$ is finite, $X - \cup_{x \in Y} U_x$ is finite. Hence $\cup_{x \in Y} U_x$ is not an element of $\mathcal{T}_\infty$. Thus when $X$ is infinite, $\mathcal{T}_\infty$ is not a topology since it is not closed under unions.

Suppose $X$ is finite. Note $\emptyset$ and $X$ are elements of $\mathcal{T}_\infty$. If $U = \emptyset$, then $X - U = X$, so $\emptyset \in \mathcal{T}_\infty$. If $U = X$, then $X - X = \emptyset$, and so $X \in \mathcal{T}_\infty$.

Let $U \in \mathcal{T}_\infty$. Since $X$ is finite, $X - U$ is not infinite. Thus $X - U = \emptyset$ or $X - U = X$. Thus, $U = X$ or $U = \emptyset$. Hence $\mathcal{T}_\infty$ is the indiscrete topology and is thus a topology.