Ph.D. Qualifying Exam in Topology

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Instructions. Do eight problems, four from each part. Some problems may require ideas from both semesters 22M:132-22M:133, and some problems may go beyond what was covered in the course. This is a closed book examination. You should have no books or papers of your own. Please do your work on the paper provided. Clearly number your pages to correspond with the problem you are working. When you start a new problem, start a new page; and please write only on one side of the paper.
You may use "big theorems" provided that the point of the problem is not the proof of the theorem.

Always justify your answers unless explicitly instructed otherwise.

Please indicate here which eight problems you want to have graded:

A1 A2 A3 A4 A5 A6

B1 B2 B3 B4 B5 B6

Notation:
$\mathbb{R}^n$ is Euclidean n-space, with the usual topology and differentiable structure.

S^n is the n-sphere, the set of points distance one from the origin in $\mathbb{R}^{n+1}$, with the subspace topology, and with the usual differentiable structure.

Part A

A: Problem 1. *Give an example of a Hausdorff space that is not metrizable. (Remember to justify your claims.)*

A: Problem 2. *Suppose* $f : \mathbb{R}^n \to \mathbb{R}$ *is continuous and for each* $x \in \mathbb{R}^n$, $f(x) \geq \|x\|$. *Prove that for each* $t \in \mathbb{R}$, $f^{-1}(t)$ *is compact.*

A: Problem 3. *Suppose* X *and* Y *are locally compact Hausdorff spaces. Prove* $X \times Y$ *is locally compact.*

A: Problem 4. *For each of these incorrect statements,*
(a) give a counter-example to the statement, and
(b) state a corrected version of the theorem. That is, modify the hypotheses so the result is a valid (nontrivial) theorem.
(You do NOT need to prove the corrected versions.)

(i) *If* $A, B \subseteq X$ *and* $f : A \cup B \to Y$ *is a function such that* $f|A$ *is continuous and* $f|B$ *is continuous, then* f *is continuous.*
(ii) *If* $f : X \to Y$ *is continuous and* $g : X \to Z$ *is a continuous surjection such that* $\forall z \in Z$, *the set* $f(g^{-1}(z))$ *is a single point* $y_z \in Y$, *then the function* $f \circ g^{-1} : Z \to Y$ *is continuous.*
(iii) *If a topological space* X *is separable, then* X *satisfies the second axiom of countability.*

A: Problem 5.

(a) Prove: If W *is a connected subset* X, *then* $\overline{W}$ *is connected.*

(b) Give an example to show that the statement,

$$W \text{ pathwise connected } \implies \overline{W} \text{ pathwise connected}$$

is not a theorem.

A: Problem 6. *Prove that a compact Hausdorff space cannot be expressed as the union of countably many nowhere-dense closed subsets.*

Part B

B: Problem 1. *Let M be an m-manifold. Let (U, ϕ) be a coordinate chart at p.*
a.) Define the standard basis $\{v_1, ..., v_m\}$ for $T_p(M)$ with respect to ϕ.
b.) Prove that $\{v_1, ..., v_m\}$ are linearly independent (note you do not need to prove they span $T_p(M)$).

B: Problem 2. *Let $\mathcal{A}$ be the maximal atlas for $\mathbb{R}$ containing $(\mathbb{R}, id)$ where id is the identity map on $\mathbb{R}$. Let $\mathcal{B}$ be the maximal atlas for $\mathbb{R}$ containing $(\mathbb{R}, f)$ where $f : \mathbb{R} \to \mathbb{R}$, $f(t) = t^3$.*
a.) Show that $\mathcal{A} \neq \mathcal{B}$.
b.) Show that $\mathbb{R}$ with the differentiable structure defined by $\mathcal{A}$ is diffeomorphic to $\mathbb{R}$ with the differentiable structure defined by $\mathcal{B}$

B: Problem 3. *Let $f : S^2 \to \mathbb{R}\mathrm{P}^2$, $f(e^{i\theta}) = e^{3i\theta}$*
a.) Define an atlas for $\mathbb{R}\mathrm{P}^2$
b.) Define an atlas for S^2.
c.) Use these atlases to show that f is smooth.
d.) Calculate $Df|_{(1,0)}$

B: Problem 4. *Let $M = \{(x, y, z) \mid x^2 + 3y - 2z - 10 = 0\}$*
a.) Prove that M is a smooth submanifold of R^3.
b.) Let $g : M \to \mathbb{R}^2$, $g(x, y, z) = (x, y)$. What are the regular values and critical values of g? Justify your answer.

B: Problem 5. *Let $\mathbb{Z}$ act on $\mathbb{R}$ by translation.*
a.) Show that $\mathbb{R}/\mathbb{Z}$ is a smooth manifold.
b.) Show that $f : \mathbb{R}^2 \to \mathbb{R}/\mathbb{Z}$, $f(x, y) = \frac{x^3y + sin(xy)}{y^2+1}$ mod 1 is smooth.
c.) Identify $\mathbb{R}/\mathbb{Z}$.

B: Problem 6. *Let V be a vector space.*
a.) Suppose $\{b_1, b_2\}$ is a basis for the vector space V. Show that $\{v_1, v_2\}$ is a basis for the same 2-dimensional subpace of V if and only if $v_1 \wedge v_2 = cb_1 \wedge b_2$ for some $c \neq 0$. What is c?

b.) State the n-dimensional analogue to part a for an n-dimensional subspace of a vector space V.