22M:132: Topology Final Exam

Dec. 17, 2008

1.) Let $\mathbf{R}$ be the set of real numbers and let $\mathbf{Z}$ be the set of integers. Let $\overline{d}(x, y) = min\{1, |x-y|\}$, Identify the following subsets of $\mathbf{R}$.

[3] 1a.) $B_{\overline{d}}(0, 1) =$ ______________

[3] 1b.) $\overline{B_{\overline{d}}(0, 1)} =$ ______________

[3] 1c.) $\{x \mid \overline{d}(x, 0) \leq 1\} =$ ______________

[3] 1d.) The set of limit points of $\mathbf{Z} = \mathbf{Z}' =$ ______________

[3] 1e.) The closure of $\mathbf{Z} = \overline{\mathbf{Z}} =$ ______________

Problems 2 and 3 are optional:

[2] 2.) An example of a paracompact space is ______________

3.) Circle T for true and F for false.

[2] 3a.) If X is paracompact, then an arbitrary union of closed sets is closed. T F

[2] 3b.) If $\mathcal{A}$ is a locally finite collection of closed subsets of X, then $\cup_{A \in \mathcal{A}} A$ is closed. T F

[80] Prove 4 from the following. Clearly indicate your choices. Note $\mathbf{R}$ is the set of real numbers

Your 4 choices: ______________

1.) Let X be a topological space in which one-point sets are closed in X. Show that X is regular if and only if for all $x \in X$, for every open set U in X such that $x \in U$, there is an open set V such that $x \in V \subset \overline{V} \subset U$.

2i.) Suppose $f : X \to Y$ is bijective and continuous, X is compact, and Y is T_2. Show that f is a homeomorphism.

ii.) Give an example of a function $f : X \to Y$ which is bijective and continuous, but not a homeomorphism where X is a subspace of a manifold and Y is a compact manifold.

3.) A connected, locally pathwise connected space is pathwise connected. (Hint: find a set which is both open and closed).

4.) Recall that if G is a topological group, then $m : G \times G \to G,\ m(x, y) = xy$ is continuous. Let G be a topological group and let $x, y \in G$.

i.) Show that for every open neighborhood U of xy, there exists open sets, V and W, such that $x \in V$, $y \in W$ and $VW \subset U$.

ii.) If U is an open set containing the identity element e, then there exists an open set V such that $e \in V$ and $V^2 = \{v_1 v_2 \mid v_i \in V\} \subset U$.

5.) Every closed subspace of a paracompact space is paracompact.

6.) Let $Y^X = \{f : X \to Y\}$. Let $S(x, U) = \{f \in Y^X \mid f(x) \in U\}$.
The topology of pointwise convergence on Y^X is the topology generated by the subbasis $\mathcal{S} = \{S(x, U) \mid x \in X, U \text{ open in } Y\}$.

i.) $S(0, (1, 2) \times (1, 2)) \subset (\mathbf{R}^2)^{\mathbf{R}}$. Give an example of a function in $S(0, (1, 2) \times (1, 2))$

ii.) Prove that the sequence f_n converges in Y^X where Y^X has the topology of pointwise convergence if and only if for all $x \in X$, the sequence $f_n(x)$ converges to $f(x)$ in Y.

7.) Suppose $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ are continuous.

i.) Show that $\{x \in X \mid f(x) = g(x)\}$ is closed in X.

ii.) Suppose $\{x \in X \mid f(x) = g(x)\}$ is dense in X (i.e., $\overline{\{x \in X \mid f(x) = g(x)\}} = X$). Show that $f = g$.

8.) Let A' = the set of limit points of A. Determine if the following statements are true.

8i.) $(A')' \subset A'$

8ii.) $A' \subset (A')'$.

9.) Every closed subspace of a compact space is compact.

Note you may choose any 4 problems from the above 9 problems (pages 2-3). Problems 1 - 6 are the same on everyone's final exam. Problems 7 - 9 are additional problems from earlier material.

Your page 1 of the exam will be worth 21 points whether or not you choose to do problems 2 and/or 3. You may do all/part/ or none of problems 2, 3 on page 1 as you choose. Problem 1 on page 1 is required.