Thm 27.1: Let X by a simply ordered set having the least upper bound property. In the order topology, each closed interval in X is compact.

Pf hint: Let $C = \{y \in (a,b] \mid [a,y]$ can be covered by a finite number of $U_\alpha\} \cup \{a\}$. Let $c = supC$.

Thm 27.3: A subspace of R^n is compact iff it is closed and bounded in the Euclidean metric or the square metric.

Idea of proof: (=>) compact Hausdorff implies closed. $\blacksquare$ For bounded, look at $A \subset \cup_{n=1}^{\infty} B(\mathbf{0}, n)$

Idea of proof: (<=) If A closed and bounded $A \subset B(\mathbf{0}, r) \subset \Pi[-r, r]$

Note: a set which is bounded in one metric can be unbounded in a different metric even when both metrics generate the same topology.

Thm 27.4 (Extreme value thm).
$f^{cont} : (X,\ compact) \to (Y,\ ordered)$ implies there exists $c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in X$.

Idea of proof: If $f(X)$ has no largest element, then $f(X) \subset \cup_{y \in f(X)}(-\infty, y)$

Defn: $f : (X, d_x) \to (Y, d_y)$ is *uniformly continuous* if for all $\epsilon > 0$, there exists a $\delta > 0$ such that $d_X(x_1, x_2) < \delta$ implies $d_Y(f(x_1), f(x_2)) < \epsilon$.

Defn: If A nonempty subset of metric space X and $x \in X$, then the *distance from x to A* is

$$d(x, A) = inf\{d(x, a) \mid a \in A\}$$

Note $d_A : X \to [0, \infty)$, $d_A(x) = d(x, A)$ is a uniformly continuous function.

Idea of proof: Show that $d(x, A) - d(y, A) \leq d(x, y)$

Defn: The diameter of $A = diam(A) =$

$$sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$$

Lemma 27.5 (Lebesgue number lemma)
Let $X \subset \cup U_\alpha$. If X is compact, there is a $\delta > 0$ such that if $diam(C) < \delta$, then there exists α_0 such that $C \subset U_{\alpha_0}$

Idea of Proof: If $X \notin \{U_\alpha \mid \alpha \in A\}$, take a finite subcover $\{U_i \mid i = 1, ..., n\}$. Let $C_i = X - U_i$.

Let $f : X \to \mathcal{R}$, $f(x) = \frac{1}{n}\Sigma d(x, C_i)$. Note f is continuous. Use extreme value thm to find $m \in X$ such that $f(m)$ is the minimum value of $f(X)$. Show $f(m) > 0$ and let $\delta = f(m)$.

Thm 27.6 (Uniform continuity thm)
$f^{cont} : (X,\ compact\ metric) \to (Y,\ metric)$ implies f uniformly continuous.

Idea of proof: $X \subset \cup_{y \in Y} f^{-1}(B_{d_Y}(y, \frac{\epsilon}{2})$. Let $\delta =$ Lebesgue number of this cover.

Defn: Given a topological space X, x is an *isolated point* of X is $\{x\}$ is open in X.

Thm 27.7: If X is a nonempty compact Hausdorff space with no isolated points, then X is uncountable.

Step 1: Take a nonempty open set U and take $x \in X$ (note x may or may not be in U).

Use Hausdorff to find nonempty open set V such that $V \subset U$ and $x \notin \overline{V}$.

Step 2: Suppose $f : \mathcal{N} \to X$, $f(x) = x_n$. Show f is not surjective (i.e., need to find a point not in the image. Which definition of compact gives us a point?).

A space, X, is *compact* if every open cover of X contains a finite subcover.

A space, X, is *compact* if for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, $\cap_{C \in \mathcal{C}} C \neq \emptyset$.

A space, X, is *limit point compact* if every infinite subset of X has a limit point.

A space, X, is *sequentially compact* if every sequence has a convergent subsequence.

Thm 28.1 Compactness implies limit point compactness, but not conversely.

Limit point compactness does not implies sequentially compactness (Hint: $((n,1))_{n=1}^{\infty}$ in $\mathcal{Z}_+ \times \{1,2\}$).

Compactness does not imply sequential compactness (Hint: $[0,1]^\omega$).

Thm 28.2: In a metrizable space X, TFAE:

1.) X is compact

2.) X is limit point compact

3.) X is sequentially compact.