22. The Quotient Topology

Defn: Let $X$ and $Y$ be topological spaces; let $p : X \to Y$ be a surjective map. The map $p$ is a **quotient map** if $U$ is open in $Y$ if and only if $p^{-1}(U)$ is open in $X$.

Defn: $C \subset X$ is **saturated** with respect to $p$ if $p^{-1}(\{y\}) \cap C \neq \emptyset$ implies $p^{-1}(\{y\}) \subset C$.

That is, $C$ is saturated if there exists a set $D \subset Y$ such that $C = p^{-1}(D)$.

That is, $C$ is saturated if $C = p^{-1}(p(C))$

Lemma: $p : X \to Y$ is a quotient map if and only if $p$ is continuous and $p$ maps saturated open sets of $X$ to open sets of $Y$.

Defn: $f : X \to Y$ is an **open map** if for every open set $U$ of $X$, $f(U)$ is open in $Y$.

Defn: $f : X \to Y$ is a **closed map** if for every closed set $A$ of $X$, $f(A)$ is closed in $Y$.

Lemma: An open map is a quotient map. A closed map is a quotient map. There exist quotient maps which are neither open nor closed.



Defn: Let $X$ be a topological spaces and let $A$ be a set; let $p : X \to Y$ be a surjective map. The **quotient topology** on $A$ is the unique topology on $A$ which makes $p$ a quotient map.

Defn: A partition, $X^*$, of a set $X$ is a collection of disjoint subsets of $X$ whose union is $X$. I.e., $X^* = \{C_\alpha \mid \alpha \in A\}$, $X^* \subset P(X)$, the power set on $X$, $C_\alpha \cap C'_\alpha = \emptyset$, for all $\alpha, \alpha' \in A$, and $X = \cup_{\alpha \in A} C_\alpha$.

Define $p : X \to X^*$, $p(x) = C_\alpha$ if $x \in C_\alpha$. The quotient topology induced by $p$ on $X^*$ is the **quotient space** of $X$. $X^*$ is called the **identification space** or decomposition space of $X$.

Thm 22.2: Let $p : X \to Y$ be a quotient map. Let $g : X \to Z$ be a map with is constant on each set $p^{-1}(y)$ for all $y \in Y$. Then $g$ induces a map $f : Y \to Z$ such that $f \circ p = g$. $f$ is continuous if and only if $g$ is continuous. $f$ is a quotient map if and only if $g$ is a quotient map.

Cor 22.3: Let $g : X \to Z$ be a surjective continuous map. Let $X^* = \{g^{-1}(\{z\} \mid z \in Z\}$ with the quotient topology.
(a.) The map $g$ induces a bijective continuous map $f : X^* \to Z$, which is a homeomorphism if and only if $g$ is a quotient map.
(b.) If $Z$ is Hausdorff, so in $X^*$.