21. Metric spaces (continued).

Lemma: If d is a metric on X and $A \subset X$, then $d|_{A\times A}$ is a metric for the subspace topology on A.

Some order topologies are metrizable, some are not.

Thm 20.3′: Suppose d_X and d_Y are metrics on X and Y, respectively. Then

$$d((x_1, y_1), (x_2, y_2)) = max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

is a metric which induces the product topology on $X \times Y$.

Note the generalization of this metric to countable products does not induce the product topology on countable products. See Thm 20.5 for a metric which does induce the product topology on R^ω.

Lemma: If X is a metric space, then X is Hausdorff.

Thm 21.2: Let $f : X \to Y$; let X and Y be metrizable with metrics d_X and d_Y, respectively. Then f is continuous if and only if for every $x \in X$ and for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$d_X(x, y) < \delta \text{ implies } d_Y(f(x), f(y)) < \epsilon.$$

Note: $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$. is equivalent to $f(B_X(x, \delta)) \subset B_Y(f(x), \epsilon)$

Lemma 21.2 (the sequence lemma): Let X be a topological space, $A \subset X$. If there exists a sequence of points in A which converge to x, then $x \in \overline{A}$

If X is metrizable, $x \in \overline{A}$ implies there exists a sequence of points in A which converge to x.

Defn: X is said to have a **countable basis at the point** x if there exists a countable collection $\mathcal{B} = \{B_n \mid n \in Z_+\}$ of neighborhoods of x such that if $x \in U^{open}$ implies there exists a $B_i \in \mathcal{B}$ such that $B_i \subset U$

X is **first countable** if X has a countable basis at each of its points.

Lemma: A metrizable space is first countable.

Lemma: If X is first countable, $x \in \overline{A}$ implies there exists a sequence of points in A which converge to x.

Thm 21.3: Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$.

Suppose X is first countable. If for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$, then f is continuous.

Lemma 21.4:

$+ : R \times R \to R$, $+(x, y) = x + y$ is continuous.

$- : R \times R \to R$, $-(x, y) = x - y$ is continuous.

$\cdot : R \times R \to R$, $\cdot(x, y) = xy$ is continuous.

$\div : R \times (R - \{0\} \to R$, $\div(x, y) = x/y$ is continuous.

Thm 21.5: If X is a topological space, and if $f, g : X \to R$ are continuous, then $f + g$, $f - g$, $f \cdot g$ are continuous. If $g(x) \neq 0$ for all x, then f/g is continuous.

Defn: Let $f_n : X \to Y$ be a sequence of functions from the set X to the topological space Y. Then the sequence of functions (f_n) **converges** to the function $f : X \to Y$ if the sequence of points $(f_n(x))$ converges to the point $f(x)$ for all $x \in X$.

Defn: Let $f_n : X \to Y$ be a sequence of functions from the set X to the metric space Y. The sequence of functions (f_n) **converges uniformly** to the function $f : X \to Y$ if for all $\epsilon > 0$, there exists an integer N such that $n > N$, $x \in X$ implies $d_Y(f_n(x), f(x)) < \epsilon$.

Thm 21.6 (uniform limit theorem):
Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If (f_n) converges uniformly to f, then f is continuous.