

17. Closed Sets and Limit Points

Defn: The set A is **closed** iff $X - A$ is open.

Thm 17.1: X be a topological space if and only if the following conditions hold:

- (1) \emptyset, X are closed.
- (2) Arbitrary intersections of closed sets are closed. ■
- (3) Finite unions of closed sets are closed.

Note arbitrary intersections of open sets need not be open. Example: $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) =$

Note arbitrary unions of closed sets need not be closed. Example: $\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1 - \frac{1}{n}\right] =$

Thm 17.2: Let Y be a subspace of X . Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y .

Thm 17.3: Let Y be a subspace of X . If A is closed in Y and Y is closed in X , then A is closed in X .

Def: The **interior** of $A = \text{Int } A = A^0 = \cup_{U^{\text{open}} \subset A} U$ ■

Def: The **closure** of $A = \text{Cl } A = \bar{A} = \cap_{A \subset F^{\text{closed}}} F$ ■

Note: \bar{A} is the smallest closed set containing A .

Thm 17.4: Let Y be a subspace of X , $A \subset Y$. Let \bar{A} denote the closure of A in X . Then the closure of A in Y equals $\bar{A} \cap Y$.

Defn: A **intersects** B if $A \cap B \neq \emptyset$

Thm 17.5: Let A be a subset of the topological space X .

(a) $x \in \bar{A}$ if and only if ($x \in U^{\text{open}}$ implies $U \cap A \neq \emptyset$).

(b) $x \in \bar{A}$ if and only if ($x \in B$ where B is a basis element implies $B \cap A \neq \emptyset$).

Defn: U is a **neighborhood** of x if U is an open set containing x .