Defn: If X is an ordered set and $a \in X$, then the following are rays in X:

$$(a, +\infty) = \{x \mid x > a\},\ (-\infty, a) = \{x \mid x < a\},$$
$$[a, +\infty) = \{x \mid x \geq a\},\ (-\infty, a] = \{x \mid x \leq a\}.$$

Lemma: The collection of all open rays is a subbasis for the order topology.

15: The Product Topology

Let $\mathcal{T}_X$ denote the topology on X and $\mathcal{T}_Y$ denote the topology on Y.

Defn: Let X and Y be topological Spaces. The **product topology** on $X \times Y$ is the topology having as basis $\mathcal{B} = \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$.

Thm 15.1: If $\mathcal{B}_X$ is a basis for the topology of X and $\mathcal{B}_Y$ is a basis for the topology of Y, then $\mathcal{D} = \{U \times V \mid U \in \mathcal{B}_X, V \in \mathcal{B}_Y\}$ is a basis for the topology of $X \times Y$.

Ex. 1: If R has the standard topology, the product topology on $R \times R$ is the standard topology on R^2.

Defn: Let $\pi_1 : X_1 \times X_2 \to X_1$, $\pi_1(x_1, x_2) = x_1$. π_1 is the projection of $X_1 \times X_2$ onto the first component.

Note: If $U \subset X_1$, then $\pi_1^{-1}(U) = U \times X_2$. Thus if U is open in X_1, then $\pi_1^{-1}(U)$ is open in $X_1 \times X_2$

Note: $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times V$

Thm 15.2: The collection

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ open in} X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in} Y\}$$

is a subbasis for the product topology on $X \times Y$.

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