

14. Order topology example:

Let $X = \{-5\} \cup (0, 1) \cup [4, 8)$. A basis for the order topology on X :

(1) All open intervals (a, b) in X .

$$(a, b) = \begin{cases} (0, b) & \text{if } a = -5, b \in (0, 1) \\ (0, 1) & \text{if } a = -5, b = 4 \\ (0, 1) \cup [4, b) & \text{if } a = -5, b \in (4, 8) \\ (a, b) & \text{if } a \in (0, 1), b \in (0, 1) \\ (a, 1) & \text{if } a \in (0, 1), b = 4 \\ (a, 1) \cup [4, b) & \text{if } a \in (0, 1), b \in (4, 8) \\ (a, b) & \text{if } a \in [4, 8), b \in (4, 8) \end{cases}$$

(2) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X .

$$[-5, b) = \begin{cases} \{-5\} \cup (0, b) & \text{if } b \in (0, 1) \\ \{-5\} \cup (0, 1) & \text{if } a = -5, b = 4 \\ \{-5\} \cup (0, 1) \cup [4, b) & \text{if } a = -5, b \in (4, 8) \end{cases}$$

(3) All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X .

X does not have a largest element hence there are no basis elements of this type.

Hence a basis for the order topology on $X = \{-5\} \cup (0, 1) \cup [4, 8)$:

$$\{(a, b) \mid (a \in [0, 1), \& b \in (0, 1]) \text{ or } (a \in [4, 8) \& b \in (4, 8))\} \cup \{(a, 1) \cup [4, b) \mid a \in [0, 1), \& b \in (4, 8)\} \cup \{\{-5\} \cup (0, b) \mid b \in (0, 1]\} \cup \{\{-5\} \cup (0, 1) \cup [4, b) \mid b \in (4, 8)\}$$

Another basis for this topology is

$$\{(a, b) \mid (a \in (0, 1), \& b \in (0, 1]) \text{ or } (a \in [4, 8) \& b \in (4, 8))\} \cup \{(a, 1) \cup [4, b) \mid a \in (0, 1), \& b \in (4, 8)\} \cup \{\{-5\} \cup (0, b) \mid b \in (0, 1)\} \cup \{\{-5\} \cup (0, 1) \cup [4, b) \mid b \in (4, 8)\}$$

Ex 2: A basis for the order topology on $Y = \{-5\} \cup [0, 1] \cup [4, 8)$:

$$\{(a, b) \mid (a \in [0, 1), \& b \in (0, 1]) \text{ or } (a \in [4, 8) \& b \in (4, 8))\} \cup \{[4, b) \mid b \in (4, 8)\} \cup \{[0, b) \mid b \in (0, 1]\} \cup \{(a, 1] \mid a \in [0, 1)\} \cup \{[0, 1]\} \cup \{5\}$$