Series Solutions Near a Regular Singular Point

MATH 365 *Ordinary Differential Equations*

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banach.millersville.edu/~bob/math365/Singular/main.pdf

Background

We will find a power series solution to the equation:

$$P(t)y'' + Q(t)y' + R(t)y = 0.$$

We will assume that t_0 is a **regular singular point.** This implies:

1. $P(t_0) = 0$,
2. $\lim_{t \to t_0} \frac{(t - t_0)Q(t)}{P(t)}$ exists,
3. $\lim_{t \to t_0} \frac{(t - t_0)^2 R(t)}{P(t)}$ exists.

Simplification

If $t_0 \neq 0$ then we can make the change of variable $x = t - t_0$ and the ODE:

$$P(x + t_0)y'' + Q(x + t_0)y' + R(x + t_0)y = 0.$$

has a regular singular point at $x = 0$.

From now on we will work with the ODE

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

having a regular singular point at $x = 0$.

Assumptions (1 of 2)

Since the ODE has a regular singular point at $x = 0$ we can define

$$x\frac{Q(x)}{P(x)} = xp(x) \quad \text{and} \quad x^2\frac{R(x)}{P(x)} = x^2q(x)$$

which are analytic at $x = 0$ and

$$\begin{aligned} \lim_{x\to 0} \frac{xQ(x)}{P(x)} &= \lim_{x\to 0} xp(x) = p_0 \\ \lim_{x\to 0} \frac{x^2R(x)}{P(x)} &= \lim_{x\to 0} x^2q(x) = q_0. \end{aligned}$$

Assumptions (2 of 2)

Furthermore since $xp(x)$ and $x^2q(x)$ are analytic,

$$\begin{aligned} xp(x) &= \sum_{n=0}^{\infty} p_n x^n \\ x^2q(x) &= \sum_{n=0}^{\infty} q_n x^n \end{aligned}$$

for all $-\rho < x < \rho$ with $\rho > 0$.

Re-writing the ODE

The second order linear homogeneous ODE can be written as

$$\begin{aligned}
0 &= P(x)y'' + Q(x)y' + R(x)y \\
&= y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y \\
&= x^2y'' + x^2\frac{Q(x)}{P(x)}y' + x^2\frac{R(x)}{P(x)}y \\
&= x^2y'' + x[xp(x)]y' + [x^2q(x)]y \\
&= x^2y'' + x\left[p_0 + p_1x + \cdots + p_nx^n + \cdots\right]y' \\
&\quad + \left[q_0 + q_1x + \cdots + q_nx^n + \cdots\right]y.
\end{aligned}$$

Special Case: Euler's Equation

If $p_n = 0$ and $q_n = 0$ for $n \geq 1$ then

$$\begin{aligned} 0 &= x^2y'' + x\left[p_0 + p_1x + \cdots + p_nx^n + \cdots\right]y' \\ &\quad + \left[q_0 + q_1x + \cdots + q_nx^n + \cdots\right]y \\ &= x^2y'' + p_0xy' + q_0y \end{aligned}$$

which is Euler's equation.

General Case

When $p_n \neq 0$ and/or $q_n \neq 0$ for some $n > 0$ then we will assume the solution to

$$x^2 y'' + x[xp(x)]y' + [x^2 q(x)]y = 0$$

has the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n},$$

an Euler solution multiplied by a power series.

Solution Procedure

Assuming $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$ we must determine:

1. the values of r,
2. a recurrence relation for a_n,
3. the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$.

Example (1 of 8)

Consider the following ODE for which $x = 0$ is a regular singular point.

$$4xy'' + 2y' + y = 0$$

Assuming $y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$ is a solution, determine the values of r and a_n for $n \geq 0$.

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} \\ y''(x) &= \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2} \end{aligned}$$

Example (2 of 8)

$$
\begin{aligned}
0 &= 4xy'' + 2y' + y \\
&= 4x \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2} + 2 \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} \\
&\quad + \sum_{n=0}^{\infty} a_n x^{r+n} \\
&= \sum_{n=0}^{\infty} 4(r+n)(r+n-1) a_n x^{r+n-1} + \sum_{n=0}^{\infty} 2(r+n) a_n x^{r+n-1} \\
&\quad + \sum_{n=0}^{\infty} a_n x^{r+n} \\
&= \sum_{n=0}^{\infty} \left[4(r+n)(r+n-1) + 2(r+n)\right] a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n}
\end{aligned}
$$

Example (3 of 8)

$$
\begin{aligned}
0 &= \sum_{n=0}^{\infty} \left[4(r+n)(r+n-1) + 2(r+n)\right] a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} \\
&= \sum_{n=0}^{\infty} 2a_n(r+n)(2r+2n-1)x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} \\
&= \sum_{n=0}^{\infty} 2a_n(r+n)(2r+2n-1)x^{r+n-1} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n-1}
\end{aligned}
$$

Example (4 of 8)

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} 2a_n(r+n)(2r+2n-1)x^{r+n-1} + \sum_{n=1}^{\infty} a_{n-1}x^{r+n-1} \\
&= 2a_0 r(2r-1)x^{r-1} + \sum_{n=1}^{\infty} 2a_n(r+n)(2r+2n-1)x^{r+n-1} \\
&\quad + \sum_{n=1}^{\infty} a_{n-1}x^{r+n-1} \\
&= 2a_0 r(2r-1)x^{r-1} + \sum_{n=1}^{\infty} \left[2a_n(r+n)(2r+2n-1) + a_{n-1}\right] x^{r+n-1}
\end{aligned}$$

Example (5 of 8)

$$\begin{aligned} 0 &= 2a_0 r(2r-1)x^{r-1} \\ &\quad + \sum_{n=1}^{\infty} \left[2a_n(r+n)(2r+2n-1) + a_{n-1}\right] x^{r+n-1} \end{aligned}$$

This implies

$$\begin{aligned} 0 &= r(2r-1) \qquad \text{(the \textbf{indicial equation}) and} \\ 0 &= 2a_n(r+n)(2r+2n-1) + a_{n-1} \end{aligned}$$

Thus we see that $r = 0$ or $r = \frac{1}{2}$ and the recurrence relation is

$$a_n = -\frac{a_{n-1}}{(2r+2n)(2r+2n-1)}, \quad \text{for } n \geq 1.$$

Example, Case $r = 0$ (6 of 8)

The recurrence relation becomes $a_n = -\dfrac{a_{n-1}}{2n(2n-1)}$.

$$\begin{aligned}
a_1 &= -\frac{a_0}{(2)(1)} = -\frac{a_0}{2!} \\
a_2 &= -\frac{a_1}{(4)(3)} = \frac{a_0}{4!} \\
a_3 &= -\frac{a_2}{(6)(5)} = -\frac{a_0}{6!} \\
&\vdots \\
a_n &= \frac{(-1)^n a_0}{(2n)!}
\end{aligned}$$

Thus $y_1(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n a_0}{(2n)!} x^{n+0} = a_0 \cos\sqrt{x}$.

Example, Case $r = 1/2$ (7 of 8)

The recurrence relation becomes $a_n = -\dfrac{a_{n-1}}{(2n+1)2n}$.

$$\begin{aligned} a_1 &= -\frac{a_0}{(3)(2)} = -\frac{a_0}{3!} \\ a_2 &= -\frac{a_1}{(5)(4)} = \frac{a_0}{5!} \\ a_3 &= -\frac{a_2}{(7)(6)} = -\frac{a_0}{7!} \\ &\vdots \\ a_n &= \frac{(-1)^n a_0}{(2n+1)!} \end{aligned}$$

Thus $y_2(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n a_0}{(2n+1)!} x^{n+1/2} = a_0 \sin \sqrt{x}$.

Example (8 of 8)

We should verify that the general solution to

$$4xy'' + 2y' + y = 0$$

is

$$y(x) = c_1 \cos \sqrt{x} + c_2 \sin \sqrt{x}.$$

Remarks

- This technique just outlined will succeed provided $r_1 \neq r_2$ and $r_1 - r_2 \neq n \in \mathbb{Z}$.
- If $r_1 = r_2$ or $r_1 - r_2 = n \in \mathbb{Z}$ then we can always find the solution corresponding to the larger of the two roots r_1 or r_2.
- The second (linearly independent) solution will have a more complicated form involving $\ln x$.

General Case: Method of Frobenius

Given $x^2y'' + x\left[xp(x)\right]y' + \left[x^2q(x)\right]y = 0$ where $x = 0$ is a regular singular point and

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n \quad \text{and} \quad x^2q(x) = \sum_{n=0}^{\infty} q_n x^n$$

are analytic at $x = 0$, we will seek a solution to the ODE of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$$

where $a_0 \neq 0$.

Substitute into the ODE

$$
\begin{aligned}
0 &= x^2 \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2} \\
&\quad + x \left[\sum_{n=0}^{\infty} p_n x^n \right] \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} + \left[\sum_{n=0}^{\infty} q_n x^n \right] \sum_{n=0}^{\infty} a_n x^{r+n} \\
&= \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n} \\
&\quad + \left[\sum_{n=0}^{\infty} p_n x^n \right] \sum_{n=0}^{\infty} (r+n) a_n x^{r+n} + \left[\sum_{n=0}^{\infty} q_n x^n \right] \sum_{n=0}^{\infty} a_n x^{r+n}
\end{aligned}
$$

Collect Like Powers of x

$$
\begin{aligned}
0 &= a_0 r(r-1)x^r + a_1(r+1)rx^{r+1} + \cdots \\
&\quad + (p_0 + p_1 x + \cdots)(a_0 r x^r + a_1(r+1)x^{r+1} + \cdots) \\
&\quad + (q_0 + q_1 x + \cdots)(a_0 x^r + a_1 x^{r+1} + \cdots) \\
&= a_0 \left[r(r-1) + p_0 r + q_0\right] x^r \\
&\quad + \left[a_1 (r+1)r + p_0 a_1 (r+1) + p_1 a_0 r + q_0 a_1 + q_1 a_0\right] x^{r+1} \\
&\quad + \cdots \\
&= a_0 \left[r(r-1) + p_0 r + q_0\right] x^r \\
&\quad + \left[a_1 \left((r+1)r + p_0(r+1) + q_0\right) + a_0 \left(p_1 r + q_1\right)\right] x^{r+1} \\
&\quad + \cdots
\end{aligned}
$$

Indicial Equation

If we define $F(r) = r(r-1) + p_0 r + q_0$ then the ODE can be written as

$$\begin{aligned} 0 &= a_0 F(r) x^r + \left[a_1 F(r+1) + a_0 \left(p_1 r + q_1\right)\right] x^{r+1} \\ &\quad + \left[a_2 F(r+2) + a_0 \left(p_2 r + q_2\right) + a_1 \left(p_1 (r+1) + q_1\right)\right] x^{r+2} \\ &\quad + \cdots \end{aligned}$$

The equation

$$0 = F(r) = r(r-1) + p_0 r + q_0$$

is called the **indicial equation.** The solutions are called the **exponents of singularity.**

Recurrence Relation

The coefficients of x^{r+n} for $n \geq 1$ determine the **recurrence relation**:

$$\begin{aligned} 0 &= a_n F(r+n) + \sum_{k=0}^{n-1} a_k \left(p_{n-k}(r+k) + q_{n-k}\right) \\ a_n &= -\frac{\sum_{k=0}^{n-1} a_k \left(p_{n-k}(r+k) + q_{n-k}\right)}{F(r+n)} \end{aligned}$$

provided $F(r+n) \neq 0$.

Exponents of Singularity

- By convention we will let the roots of the indicial equation $F(r) = 0$ be r_1 and r_2.
- When r_1 and $r_2 \in \mathbb{R}$ we will assign subscripts so that $r_1 \geq r_2$.
- Consequently the recurrence relation where $r = r_1$,

$$a_n(r_1) = -\frac{\sum_{k=0}^{n-1} a_k \left(p_{n-k}(r_1 + k) + q_{n-k}\right)}{F(r_1 + n)}$$

  is defined for all $n \geq 1$.
- One solution to the ODE is then

$$y_1(x) = x^{r_1} \left(1 + \sum_{n=1}^{\infty} a_n(r_1) x^n\right).$$

Case: $r_1 - r_2 \notin \mathbb{N}$

- If $r_1 - r_2 \neq n$ for any $n \in \mathbb{N}$ then $r_1 \neq r_2 + n$ for any $n \in \mathbb{N}$ and consequently $F(r_2 + n) \neq 0$ for any $n \in \mathbb{N}$.
- Consequently the recurrence relation where $r = r_2$,

$$a_n(r_2) = -\frac{\sum_{k=0}^{n-1} a_k \left(p_{n-k}(r_2 + k) + q_{n-k}\right)}{F(r_2 + n)}$$

  is defined for all $n \geq 1$.
- A second solution to the ODE is then

$$y_2(x) = x^{r_2}\left(1 + \sum_{n=1}^{\infty} a_n(r_2)x^n\right).$$

Example

Find the indicial equation, exponents of singularity, and discuss the nature of solutions to the ODE

$$x^2y'' - x(2+x)y' + (2+x^2)y = 0$$

near the regular singular point $x = 0$.

Solution

$$\begin{aligned} p_0 &= \lim_{x\to 0} x\frac{-x(2+x)}{x^2} = -\lim_{x\to 0}(2+x) = -2 \\ q_0 &= \lim_{x\to 0} x^2\frac{2+x^2}{x^2} = \lim_{x\to 0}(2+x^2) = 2 \end{aligned}$$

The indicial equation is then

$$\begin{aligned} r(r-1)+(-2)r+2 &= 0 \\ r^2-3r+2 &= 0 \\ (r-2)(r-1) &= 0. \end{aligned}$$

The exponents of singularity are $r_1 = 2$ and $r_2 = 1$. Consequently we have one solution of the form

$$y_1(x) = x^2\left(1+\sum_{n=1}^{\infty} a_n x^n\right).$$

Case: $r_1 = r_2$ Equal Exponents of Singularity (1 of 4)

- When the exponents of singularity are equal then $F(r) = (r - r_1)^2$.
- We have a solution to the ODE of the form

$$y_1(x) = x^r \left(1 + \sum_{n=1}^{\infty} a_n(r) x^n\right).$$

- Differentiating this solution and substituting into the ODE yields

$$\begin{aligned} 0 &= a_0 F(r) x^r \\ &\quad + \sum_{n=1}^{\infty} \left[a_n F(r+n) + \sum_{k=0}^{n-1} a_k \left(p_{n-k}(r+k) + q_{n-k}\right) \right] x^{r+n} \\ &= a_0 (r - r_1)^2 x^r. \end{aligned}$$

when a_n solves the recurrence relation.

Case: $r_1 = r_2$ Equal Exponents of Singularity (2 of 4)

Recall: for the ODE $x^2 y'' + x[xp(x)]y' + [x^2 q(x)]y = 0$ we can define the **linear operator**

$$L[y] = x^2 y'' + x[xp(x)]y' + [x^2 q(x)]y$$

so that the ODE can be written compactly as $L[y] = 0$.

Consider the infinite series solution to the ODE,

$$\phi(r,x) = x^r \left[1 + \sum_{n=1}^{\infty} a_n(r) x^n\right].$$

Note: since the coefficients of the series depend on r we denote the solution as $\phi(r,x)$.

Case: $r_1 = r_2$ Equal Exponents of Singularity (3 of 4)

$$
\begin{aligned}
0 &= L[\phi](r_1, x) \\
0 &= a_0 (r - r_1)^2 x^r \Big|_{r=r_1} \\
\frac{\partial}{\partial r}(0)\Big|_{r=r_1} &= \frac{\partial}{\partial r}\left(a_0 (r - r_1)^2 x^r \right)\Big|_{r=r_1} \\
0 &= 2a_0 (r - r_1) x^r \big|_{r=r_1} + a_0 (r - r_1)^2 (\ln x) x^r \Big|_{r=r_1} \\
0 &= a_0 (r - r_1)^2 (\ln x) x^r \Big|_{r=r_1} \\
0 &= L\left[\frac{\partial \phi}{\partial r}\right](r_1, x)
\end{aligned}
$$

Thus a second solution to the ODE is $y_2(x) = \left.\dfrac{\partial \phi(r, x)}{\partial r}\right|_{r=r_1}$.

Case: $r_1 = r_2$ Equal Exponents of Singularity (4 of 4)

$$
\begin{aligned}
y_2(x) &= \left.\frac{\partial \phi(r,x)}{\partial r}\right|_{r=r_1} \\
&= \left.\frac{\partial}{\partial r}\left(x^r\left[1+\sum_{n=1}^{\infty} a_n(r)x^n\right]\right)\right|_{r=r_1} \\
&= \left.(\ln x)x^r\left[1+\sum_{n=1}^{\infty} a_n(r)x^n\right] + x^r\sum_{n=1}^{\infty} a_n'(r)x^n\right|_{r=r_1} \\
&= (\ln x)y_1(x) + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n
\end{aligned}
$$

Example (1 of 9)

Find the general solution to the ODE:

$$xy'' + y' + xy = 0$$

near the regular singular point $x = 0$.

$$\begin{aligned} \lim_{x \to 0} x \left(\frac{1}{x} \right) &= 1 = p_0 \\ \lim_{x \to 0} x^2 \left(\frac{x}{x} \right) &= 0 = q_0 \end{aligned}$$

Thus the indicial equation is $F(r) = r(r-1) + r = r^2 = 0$ and the exponents of singularity are $r_1 = r_2 = 0$.

Example (2 of 9)

Assume $y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$, differentiate, and substitute into the given ODE.

$$
\begin{aligned}
0 &= x \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2} + \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} \\
&\quad + x \sum_{n=0}^{\infty} a_n x^{r+n} \\
&= \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-1} + \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} \\
&\quad + \sum_{n=0}^{\infty} a_n x^{r+n+1} \\
&= \sum_{n=0}^{\infty} (r+n)^2 a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n+1}
\end{aligned}
$$

Example (3 of 9)

$$
\begin{aligned}
0 &= \sum_{n=0}^{\infty}(r+n)^2 a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n+1} \\
&= \sum_{n=0}^{\infty}(r+n)^2 a_n x^{r+n-1} + \sum_{n=2}^{\infty} a_{n-2} x^{r+n-1} \\
&= a_0 r^2 x^{r-1} + a_1 (r+1)^2 x^r + \sum_{n=2}^{\infty}(r+n)^2 a_n x^{r+n-1} \\
&\quad + \sum_{n=2}^{\infty} a_{n-2} x^{r+n-1} \\
&= a_0 r^2 x^{r-1} + a_1 (r+1)^2 x^r + \sum_{n=2}^{\infty} \left[(r+n)^2 a_n + a_{n-2} \right] x^{r+n-1}
\end{aligned}
$$

Example (4 of 9)

$$0 = a_0 r^2 x^{r-1} + a_1 (r+1)^2 x^r + \sum_{n=2}^{\infty} \left[(r+n)^2 a_n + a_{n-2} \right] x^{r+n-1}$$

- The exponents of singularity are $r_1 = r_2 = 0$.
- The recurrence relation is $a_n(r) = -\dfrac{a_{n-2}(r)}{(r+n)^2}$.
- $a_1 = 0$ which implies $a_{2n+1} = 0$ for all $n \in \mathbb{N}$.

Example (5 of 9)

When $r = 0$, and a_0 is arbitrary

$$
\begin{aligned}
a_2 &= -\frac{a_0}{2^2} = -\frac{a_0}{4^1(1!)^2} \\
a_4 &= -\frac{a_2}{4^2} = \frac{a_0}{4^2(2!)^2} \\
a_6 &= -\frac{a_4}{6^2} = -\frac{a_0}{4^3(3!)^2} \\
&\vdots \\
a_{2n} &= \frac{(-1)^n a_0}{4^n(n!)^2}
\end{aligned}
$$

thus

$$y_1(x) = a_0 \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{4^n (n!)^2} \right).$$

Example (6 of 9)

Now find the second solution.

$$
\begin{aligned}
a_n(r) &= -\frac{a_{n-2}(r)}{(r+n)^2} \\
a_n'(r) &= -\frac{a_{n-2}'(r)(r+n)^2 - a_{n-2}(r)2(r+n)}{(r+n)^4} \\
&= -\frac{a_{n-2}'(r)(r+n) - 2a_{n-2}(r)}{(r+n)^3} \\
a_n'(0) &= \frac{2a_{n-2}(0) - na_{n-2}'(0)}{n^3}
\end{aligned}
$$

Example (7 of 9)

Since $a_{2n+1}(r) = 0$ for all $n \in \mathbb{N}$ then $a'_{2n+1}(r) = 0$ for all $n \in \mathbb{N}$.

Since a_0 is an arbitrary **constant** then $a'_0 = 0$.

Example (8 of 9)

Recall the recurrence relation for $n \geq 2$:

$$a_n'(0) = \frac{2a_{n-2}(0) - n a_{n-2}'(0)}{n^3}$$

If $n = 2$ then

$$\begin{aligned} a_2'(0) &= \frac{2a_0 - 2a_0'}{2^3} \\ &= \frac{a_0}{4} = (1)\frac{a_0}{4^1(1!)^2} \end{aligned}$$

If $n = 4$ then

$$\begin{aligned} a_4'(0) &= \frac{2a_2 - 4a_2'}{4^3} \\ &= \frac{a_2 - 2a_2'}{4^2(2!)} \\ &= \frac{1}{4^2(2!)}\left(-\frac{a_0}{4} - 2\left(\frac{a_0}{4}\right)\right) \\ &= -\left(1 + \frac{1}{2}\right)\frac{a_0}{4^2(2!)^2} \end{aligned}$$

$$\begin{aligned}
a_6'(0) &= \frac{2a_4 - 6a_4'}{6^3} \\
&= \frac{a_4 - 3a_4'}{6^2(3)} \\
&= \left(1 + \frac{1}{2} + \frac{1}{3}\right) \frac{a_0}{4^3(3!)^2} \\
&\vdots \\
a_{2n}'(0) &= \frac{(-1)^{n+1} \sum_{k=1}^{n} \frac{1}{k}}{4^n(n!)^2}
\end{aligned}$$

Thus

$$y_2(x) = (\ln x)y_1(x) + \sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1} \sum_{k=1}^{n} \frac{1}{k}}{4^n(n!)^2} \right) x^{2n}.$$

The Story So Far (1 of 3)

Considering the second-order linear, homogeneous ODE:

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

where $x_0 = 0$ is a regular singular point.

This implies $P(x_0) = 0$ and

$$\begin{aligned}
\lim_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)} &= \lim_{x \to 0} x\, p(x) = p_0 \\
\lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)} &= \lim_{x \to 0} x^2 q(x) = q_0.
\end{aligned}$$

The Story So Far (2 of 3)

Define the polynomial $F(r) = r(r-1) + p_0 r + q_0$, then

$$r(r-1) + p_0 r + q_0 = 0$$

is called the **indicial equation** and the roots $r_1 \geq r_2$ are called the **exponents of singularity**.

The Story So Far (2 of 3)

Define the polynomial $F(r) = r(r-1) + p_0 r + q_0$, then

$$r(r-1) + p_0 r + q_0 = 0$$

is called the **indicial equation** and the roots $r_1 \geq r_2$ are called the **exponents of singularity.**

If $r_1 - r_1 \notin \mathbb{N}$ then we have a fundamental set of solutions of the form

$$\begin{aligned} y_1(x) &= x^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right] \\ y_2(x) &= x^{r_2}\left[1 + \sum_{n=1}^{\infty} a_n(r_2)x^n\right]. \end{aligned}$$

The Story So Far (3 of 3)

If $r_1 = r_2$ then we have a fundamental set of solutions of the form

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right] \\
y_2(x) &= y_1(x)\ln x + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n.
\end{aligned}
$$

The Story So Far (3 of 3)

If $r_1 = r_2$ then we have a fundamental set of solutions of the form

$$\begin{aligned} y_1(x) &= x^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right] \\ y_2(x) &= y_1(x)\ln x + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n. \end{aligned}$$

Now we may take up the final case when $r_1 - r_2 \in \mathbb{N}$.

Case: $r_1 - r_2 = N \in \mathbb{N}$

The second solution has the form

$$\begin{aligned} y_2(x) &= a\, y_1(x) \ln x + x^{r_2} \left[1 + \sum_{n=1}^{\infty} c_n(r_2) x^n \right] \quad \text{where} \\ a &= \lim_{r \to r_2} (r - r_2) a_N(r) \quad \text{and} \\ c_n(r_2) &= \left. \frac{d}{dr} \left[(r - r_2) a_n(r) \right] \right|_{r=r_2} . \end{aligned}$$

We can assume $a_0 = 1$ for simplicity.

Example (1 of 8)

Find the general solution to the ODE

$$x\,y'' - y = 0$$

with regular singular point at $x = 0$.

$$\begin{aligned} \lim_{x\to 0} x\left(\frac{0}{x}\right) &= 0 = p_0 \\ \lim_{x\to 0} x^2\left(\frac{-1}{x}\right) &= 0 = q_0 \end{aligned}$$

Thus the indicial equation is $F(r) = r(r-1)$ and the exponents of singularity are $r_1 = 1$ and $r_2 = 0$.

Example (2 of 8)

$$
\begin{aligned}
0 &= x\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-2} - \sum_{n=0}^{\infty} a_n x^{r+n} \\
&= \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-1} - \sum_{n=0}^{\infty} a_n x^{r+n} \\
&= \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-1} - \sum_{n=1}^{\infty} a_{n-1} x^{r+n-1} \\
&= a_0 r(r-1)x^{r-1} + \sum_{n=1}^{\infty}\left[(r+n)(r+n-1)a_n - a_{n-1}\right] x^{r+n-1}
\end{aligned}
$$

Example (3 of 8)

Recurrence relation for $n \geq 1$:

$$\begin{aligned} a_n(r) &= \frac{a_{n-1}(r)}{(r+n)(r+n-1)} \\ a_n(1) &= \frac{a_{n-1}(1)}{n(n+1)} \end{aligned}$$

If $a_0 = 1$ then

$$a_n(1) = \frac{1}{n!(n+1)!}$$

and

$$y_1(x) = x\left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n!(n+1)!}\right].$$

Example (4 of 8)

According to the formula of Frobenius

$$y_2(x) = ay_1(x)\ln x + x^{r_2}\left[1 + \sum_{n=1}^{\infty} c_n(r_2)x^n\right].$$

$$\begin{aligned} a &= \lim_{r\to r_2}(r - r_2)a_N(r) \\ &= \lim_{r\to 0} ra_1(r) \\ &= \lim_{r\to 0} r\,\frac{a_0}{r(r+1)} \\ &= \lim_{r\to 0}\frac{1}{r+1} \\ &= 1 \end{aligned}$$

Example (5 of 8)

$$\begin{aligned}
c_1(r_2) &= \frac{d}{dr}\left[(r - r_2)a_1(r)\right]\Big|_{r=r_2} \\
c_1(0) &= \frac{d}{dr}\left[\frac{ra_0}{r(r+1)}\right]\Bigg|_{r=0} \\
&= \frac{d}{dr}\left[\frac{a_0}{r+1}\right]\Bigg|_{r=0} \\
&= \frac{d}{dr}\left[\frac{1}{r+1}\right]\Bigg|_{r=0} \\
&= -1
\end{aligned}$$

Example (6 of 8)

$$
\begin{aligned}
c_2(r_2) &= \frac{d}{dr}\left[(r - r_2)a_2(r)\right]\Big|_{r=r_2} \\
c_2(0) &= \frac{d}{dr}\left[r a_2(r)\right]\Big|_{r=0} \\
&= \frac{d}{dr}\left[\frac{r a_1(r)}{(r+1)(r+2)}\right]\Bigg|_{r=0} \\
&= \frac{d}{dr}\left[\frac{r a_0}{r(r+1)^2(r+2)}\right]\Bigg|_{r=0} \\
&= \frac{d}{dr}\left[\frac{1}{(r+1)^2(r+2)}\right]\Bigg|_{r=0} \\
&= -\frac{5}{4}
\end{aligned}
$$

Example (7 of 8)

$$
\begin{aligned}
c_3(r_2) &= \frac{d}{dr}\left[(r - r_2)a_3(r)\right]\Big|_{r=r_2} \\
c_3(0) &= \frac{d}{dr}\left[\frac{1}{(r+1)^2(r+2)^2(r+3)}\right]\Bigg|_{r=0} \\
&= -\frac{5}{18}
\end{aligned}
$$

Example (8 of 8)

So the second solution has the form

$$y_2(x) = y_1(x) \ln x + 1 - x - \frac{5}{4}x^2 - \frac{5}{18}x^3 + \cdots .$$