

Note that $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$ and $A(c\mathbf{x}) = cA\mathbf{x}$

A system of equations is $A\mathbf{x} = \mathbf{b}$ is **homogeneous** if $\mathbf{b} = \mathbf{0}$.

Suppose $A\mathbf{u} = \mathbf{0}$, $A\mathbf{v} = \mathbf{0}$, and $A\mathbf{p} = \mathbf{b}$, then

$$\begin{aligned} A(c_1\mathbf{u} + c_2\mathbf{v} + \mathbf{p}) &= c_1A\mathbf{u} + c_2A\mathbf{v} + A\mathbf{p} \\ &= c_1(\mathbf{0}) + c_2(\mathbf{0}) + \mathbf{b} = \mathbf{b} \end{aligned}$$

I.e., $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v} + \mathbf{p}$ is a soln to $A\mathbf{x} = \mathbf{b}$ for any c_1, c_2 .

Solve the following systems of equations:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 2 \\ 4 & 5 & 6 & 0 & 3 & 5 \\ 7 & 8 & 9 & 0 & 0 & 8 \end{bmatrix}$$

$$\downarrow R_2 - 4R_1 \rightarrow R_2, \quad R_3 - 7R_1 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 2 \\ 0 & -3 & -6 & 0 & 3 & -3 \\ 0 & -6 & -12 & -7 & 0 & -6 \end{bmatrix}$$

$$\downarrow R_3 - 2R_1 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 2 \\ 0 & -3 & -6 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 & -6 & 0 \end{bmatrix}$$

\downarrow already know sol'n to system b.

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 2 \\ 0 & -3 & -6 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\downarrow -\frac{1}{3}R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 2R_2 \rightarrow R_1} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \quad \text{no solution}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Check:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \& \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

Compare to solving linear homogeneous differential eqn:

Ex: $ay'' + by' + cy = g(t)$

1.) Easily solve homogeneous DE: $ay'' + by' + cy = 0$

$y = e^{rt} \Rightarrow ar^2 + br + c = 0 \Rightarrow y = c_1\phi_1 + c_2\phi_2$ for homogeneous solution (see sections 3.1, 3.3, 3.4).

2.) More work: Find one solution to $ay'' + by' + cy = g(t)$ (see sections 3.5, 3.6)

If $y = \psi(t)$ is a soln, then general soln to $ay'' + by' + cy = g(t)$ is

$$y = c_1\phi_1 + c_2\phi_2 + \psi$$

Check: $a\phi_1'' + b\phi_1' + c\phi_1 = 0$

$$a\phi_2'' + b\phi_2' + c\phi_2 = 0$$

$$a\psi'' + b\psi' + c\psi = g(t)$$

To solve $ay'' + by' + cy = g_1(t) + g_2(t)$

1.) Solve $ay'' + by' + cy = 0 \Rightarrow y = c_1\phi_1 + c_2\phi_2$ for homogeneous solution.

2a.) Solve $ay'' + by' + cy = g_1(t) \Rightarrow y = \psi_1$

2b.) Solve $ay'' + by' + cy = g_2(t) \Rightarrow y = \psi_2$

General solution to $ay'' + by' + cy = g_1(t) + g_2(t)$ is

$$y = c_1\phi_1 + c_2\phi_2 + \psi_1 + \psi_2$$