Note: You must be able to identify which techniques you need to use. For example:

Integration:

* Integration by substitution

* Integration by parts

* Integration by partial fractions

Note: Partial fractions are also used in ch 6 for a different application.

For differential equations:

Is the differential equation 1rst order or 2nd order?

If 2nd order: Section 3.1, solve $ay'' + by' + cy = 0$.

Guess $y = e^{rt}$.
$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,
Need to have two independent solutions.

If $y = \phi_1, y = \phi_2$ are solutions to a LINEAR HOMOGENEO
differential equation, $y = c_1\phi_1 + c_2\phi_2$ is also a solution

If 1st order: Is the equation linear or separable or ?

Solving first order differential equation:

Method 1 (sect. 2.2): Separate variables.

Method 2 (sect. 2.1): If linear $[y'(t)+p(t)y(t) = g(t)]$, multiply equation by an integrating factor
$u(t) = e^{\int p(t)dt}$.

$$
\begin{gathered}
y' + py = g \\
y'u + upy = ug \\
(uy)' = ug \\
\int (uy)' = \int ug \\
uy = \int ug \\
\text{etc...}
\end{gathered}
$$

Method 3 (sect. 2.4): Solve Bernoulli's equation,

$$y' + p(t)y = g(t)y^n,$$

when $n > 1$ by changing it to a linear equation by substituting $v = y^{1-n}$

---

direction field = slope field = graph of $\frac{dv}{dt}$ in t, v-plane.

*** can use slope field to determine behavior of v including as $t \to \infty$.
Equilibrium Solution = constant solution

stable, unstable, semi-stable.

Section 2.4: Existence and Uniqueness.

In general, for $y' = f(t, y), \;\; y(t_0) = y_0$**,**
solution may or may not exist and solution may or may not be unique.

But we have 2 theorems that guarantee both existence and uniqueness of solutions under certain conditions:

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a, b) \to R$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$y' + p(t)y = g(t),$$
$$y(t_0) = y_0$$

1st order differential equation (general case):

Thm 2.4.2: Suppose $z = f(t, y)$ and $z = \frac{\partial f}{\partial y}(t, y)$ are continuous on $(a, b) \times (c, d)$ and the point $(t_0, y_0) \in (a, b) \times (c, d)$, then there exists an interval $(t_0 - h, t_0 + h) \subset (a, b)$ such that there exists a unique function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

$$y' = f(t, y), \;\; y(t_0) = y_0.$$

Note the initial value problem

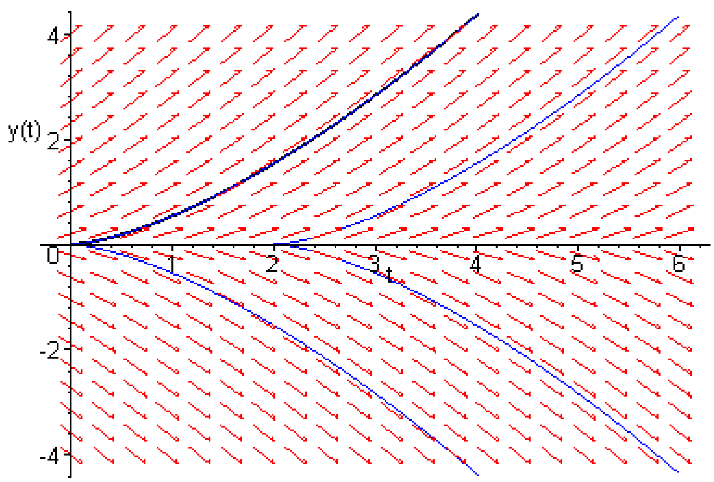
$$y' = y^{\frac{1}{3}}, \ y(0) = 0$$

has an infinite number of different solutions.

$$y^{-\frac{1}{3}} dy = dt$$
$$\frac{3}{2} y^{\frac{2}{3}} = t + C$$
$$y = \pm(\frac{2}{3}t + C)^{\frac{3}{2}}$$
$$y(0) = 0 \text{ implies } C = 0$$

Thus $y = \pm(\frac{2}{3}t)^{\frac{3}{2}}$ are solutions.

$y = 0$ is also a solution, etc.



Compare to Thm 2.4.2:
$f(t, y) = y^{\frac{1}{3}}$ is continuous near (0, 0)
But $\frac{\partial f}{\partial y}(t, y) = \frac{1}{3} y^{\frac{-2}{3}}$ is not continuous near (0, 0) since it isn't defined at (0, 0).

Section 2.4 example: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$

$F(y,t) = \frac{1}{(1-t)(2-y)}$ is continuous for all $t \neq 1$, $y \neq 2$

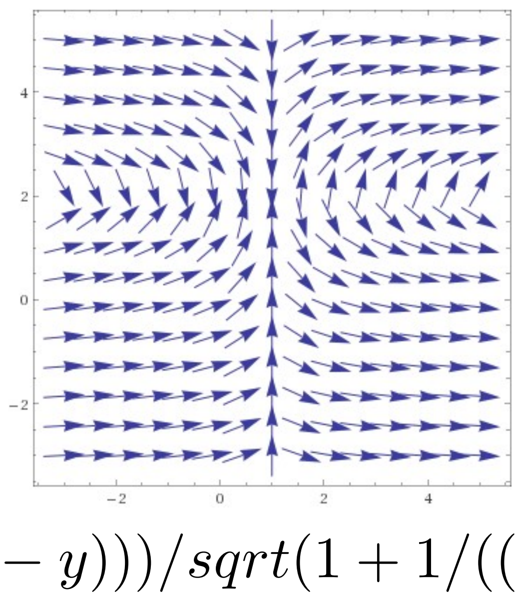
$$\frac{\partial F}{\partial y} = \frac{\partial\left(\frac{1}{(1-t)(2-y)}\right)}{\partial y} = \frac{1}{(1-t)}\frac{\partial(2-y)^{-1}}{\partial y} = \frac{1}{(1-t)(2-y)^2}$$

$\frac{\partial F}{\partial y}$ is continuous for all $t \neq 1$, $y \neq 2$

Thus the IVP $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = y_0$ has a unique solution if $t_0 \neq 1$, $y_0 \neq 2$.

Note that if $y_0 = 2$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = 2$ has two solutions if $t_0 \neq 1$ (and if we allow vertical slope in domain. Note normally our convention will be to NOT allow vertical slope in domain of solution).

Note that if $t_0 = 1$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(1) = y_0$ has no solutions.



$(1, 1/((1-t)(2-y)))/sqrt(1 + 1/((1-t)(2-y))^2)$

Solve via separation of variables: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$

$\int(2-y)dy = \int \frac{dt}{1-t}$ implies $2y - \frac{y^2}{2} = -ln|1-t| + C$

$y^2 - 4y - 2ln|1-t| + C = 0$

$y = \frac{4 \pm \sqrt{16+4(2ln|1-t|+C)}}{2} = 2 \pm \sqrt{4 + 2ln|1-t| + C}$

$$y = 2 \pm \sqrt{2ln|1-t| + C}$$

Find domain: $2ln|1-t| + C \geq 0$ & $t \neq 1$ & $y \neq 2$

NOTE: the convention in this class to to choose largest possible connected domain where tangent line to solution is never vertical.

$2ln|1-t| \geq -C$ and $t \neq 1$ and $y \neq 2$ implies

$ln|1-t| > -\frac{C}{2}$ Note: we want to find domain for this C and thus this C can't swallow constants).

$|1-t| > e^{-\frac{C}{2}}$ since e^x is an increasing function.

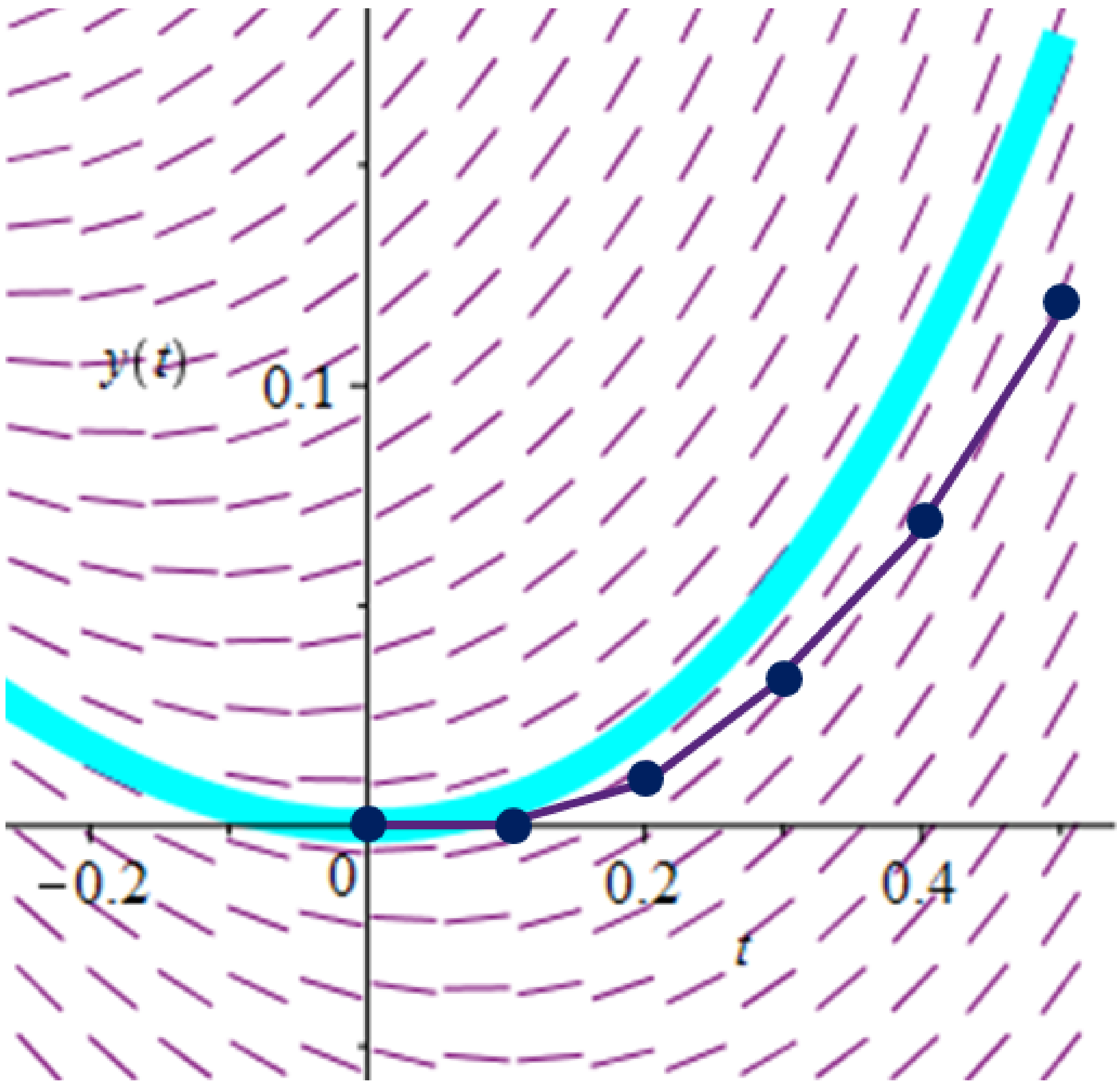
$1-t < -e^{-\frac{C}{2}}$ or $1-t > e^{-\frac{C}{2}}$

Domain: $\begin{cases} t > e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 1 \\ t < -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 1. \end{cases}$

2.7: Approximating soln to IVP using multiple tangent lines.

Example: $y' = t + 2y$, $y(0) = 0$

$$y(t) = \begin{cases} 0 & 0 \leq t \leq 0.1 \\ 0.1t - 0.01 & 0.1 \leq t \leq 0.2 \\ 0.22t - 0.034 & 0.2 \leq t \leq 0.3 \\ 0.364t - 0.0772 & 0.3 \leq t \leq 0.4 \\ 0.5328t - 0.14672 & 0.4 \leq t \leq 0.5 \end{cases}$$



2.8: Approximating soln to IVP using seq of fns,

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds$$

Example: $y' = t + 2y$, $y(0) = 0$

$\phi_0(t) = 0$, $\phi_1(t) = \frac{t^2}{2}$, $\phi_2(t) = \frac{t^2}{2} + \frac{t^3}{3}$,

$\phi_3(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$, $\phi_4(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6} + \frac{t^5}{15}$