

Recall that a constant solution is an equilibrium solution. Thus its derivative is 0.

To find an equilibrium solution (i.e., constant solution), plug it in (for example, plug in $y(t) = k$ or $x_1(t) = k_1, x_2(t) = k_2$ depending on variables used and if you have one DE or a system of two DEs). Since the derivative of a constant is zero, this is equivalent to setting the derivative = 0.

Find all equilibrium solutions and classify them (stable, asymptotically stable, semi-stable, unstable and if system of DEs, node, saddle, spiral, center). In the case of non-linear system of DEs, state all possibilities for type of equilibrium solution.

If the (system of) differential equation(s) does not have an equilibrium solution, state so (note 4 of the following 16 problems below do not have an equilibrium solution).

Hint: The eigenvalues of upper and lower triangular matrices are the diagonal entries.

Note: You do not need to draw any direction fields.

1.) $y' = (y - 3)^4(y - 5)^9$ $y = 3$ is semi-stable, $y = 5$ is unstable.

2.) $y' = y^2 + 2$ no equilibrium solution.

3.) $y' = \sin(y)$ $y = 2n\pi$ is unstable, $y = (2n + 1)\pi$ is asymptotically stable.

4.) $y' = \sin(t)$ no equilibrium solution.

5.) $y' = \sin^2(y)$ $y = n\pi$ is semi-stable.

6.) $y' = \sin^2(t)$ no equilibrium solution.

7.) $y' = ty$ $y = 0$ is unstable.

8.) $x' = 4 - y^2, y' = (x + 1)(y - x)$

If $4 - y^2 = 0$, then $y = \pm 2$

If $y = 2$, then $(x + 1)(y - x) = (x + 1)(2 - x) = 0$. Thus $x = -1, 2$.

If $y = -2$, then $(x + 1)(y - x) = (x + 1)(-2 - x) = 0$. Thus $x = -1, -2$.

Jacobian matrix:
$$\begin{bmatrix} 0 & -2y \\ y - 2x - 1 & x + 1 \end{bmatrix}$$

For $(x, y) = (-1, 2)$, Jacobian matrix is
$$\begin{bmatrix} 0 & -4 \\ 3 & 0 \end{bmatrix}$$

Thus $(x(t), y(t)) = (-1, 2)$ is a stable center or unstable spiral or asymptotically stable spiral.

For $(x, y) = (2, 2)$, Jacobian matrix is
$$\begin{bmatrix} 0 & -4 \\ -3 & 3 \end{bmatrix}$$

Thus $(x(t), y(t)) = (2, 2)$ is an unstable saddle.

For $(x, y) = (-1, -2)$, Jacobian matrix is $\begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}$

Thus $(x(t), y(t)) = (-1, -2)$ is a stable center or unstable spiral or asymptotically stable spiral.

For $(x, y) = (-2, -2)$, Jacobian matrix is $\begin{bmatrix} 0 & 4 \\ 1 & -1 \end{bmatrix}$

Thus $(x(t), y(t)) = (-2, -2)$ is an unstable saddle.

9.) $x' = x - 2, y' = x - 1$ no equilibrium solution.

$$10.) \mathbf{x}' = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}$$

One positive (1) and one negative eigenvalue (-2). Thus $(x_1(t), x_2(t)) = (0, 0)$ is an unstable saddle.

$$11.) \mathbf{x}' = \begin{bmatrix} 1 & 0 \\ 5 & -2 \end{bmatrix} \mathbf{x}$$

One positive (1) and one negative eigenvalue (-2). Thus $(x_1(t), x_2(t)) = (0, 0)$ is an unstable saddle.

$$12.) \mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}$$

Purely imaginary eigenvalues $i, -i$. Thus $(x_1(t), x_2(t)) = (0, 0)$ is a stable center.

$$13.) \mathbf{x}' = \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \mathbf{x}$$

Two positive eigenvalues 1, 2. Thus $(x_1(t), x_2(t)) = (0, 0)$ is an unstable node.

$$14.) \mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \mathbf{x}$$

Two complex eigenvalues, $-1 \pm 2i$, with negative real part. Thus $(x_1(t), x_2(t)) = (0, 0)$ is an asymptotically stable spiral.

$$15.) \mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -5 & 2 \end{bmatrix} \mathbf{x}$$

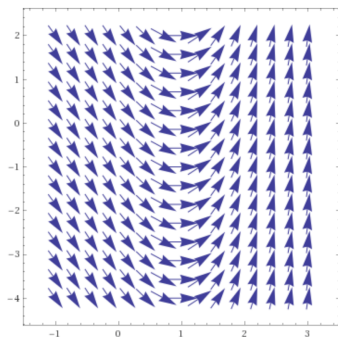
Two complex eigenvalues, $1 \pm 2i$, with positive real part. Thus $(x_1(t), x_2(t)) = (0, 0)$ is an unstable spiral.

$$16.) \mathbf{x}' = \begin{bmatrix} -1 & 0 \\ 5 & -2 \end{bmatrix} \mathbf{x}$$

Two negative eigenvalues -1, -2. Thus $(x_1(t), x_2(t)) = (0, 0)$ is an asymptotically stable node.

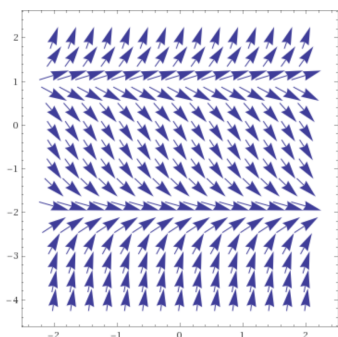
Problems 17 - 20 show the slope field for a first order differential equations. In addition to determining and classifying all equilibrium solutions (if any), also draw the trajectories satisfying the initial values $y(0) = 1$, $y(1) = 0$, $y(1) = 2$, $y(0) = -3$.

17.)



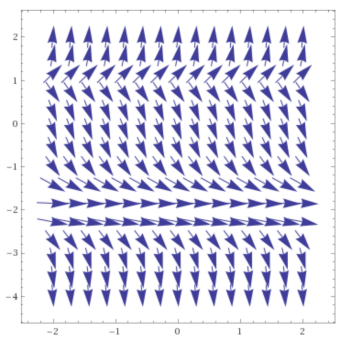
No equilibrium solution.

18.)



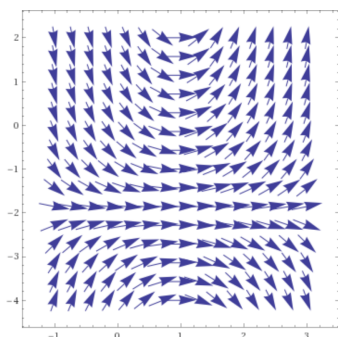
$y = 1$ is unstable. $y = -2$ is asymptotically stable.

19.)



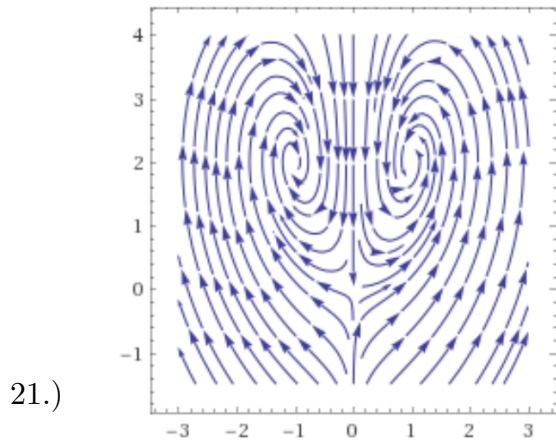
$y = 1$ is unstable. $y = -2$ is semi-stable.

20.)



$y = -2$ is unstable.

Problems 21-23 show the stream plot in the $x_1 - x_2$ -plane for a system of two first order differential equations. In addition to determining and classifying all equilibrium solutions, also draw the trajectories satisfying the initial values $(x_1(0), x_2(0)) = (0, 1)$, $(x_1(0), x_2(0)) = (1, 0)$, $(x_1(0), x_2(0)) = (1, 2)$, $(x_1(0), x_2(0)) = (-1, 0)$. Also describe the basins of attraction.



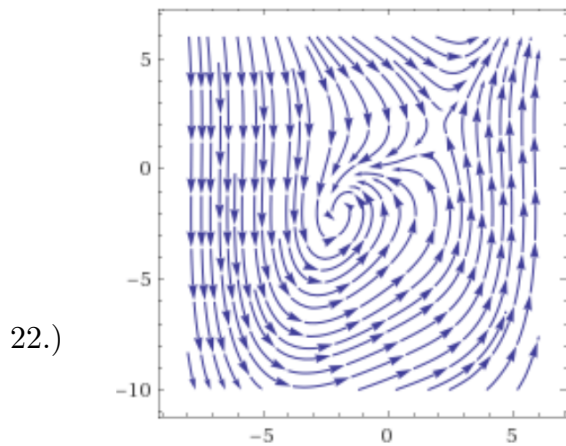
$(x_1(t), x_2(t)) = (0, 0)$ is an unstable saddle.

$(x_1(t), x_2(t)) = (1, 2)$ is an asymptotically stable node.

basin of attraction: $x_1 > 0$.

$(x_1(t), x_2(t)) = (-1, 2)$ is an asymptotically stable node.

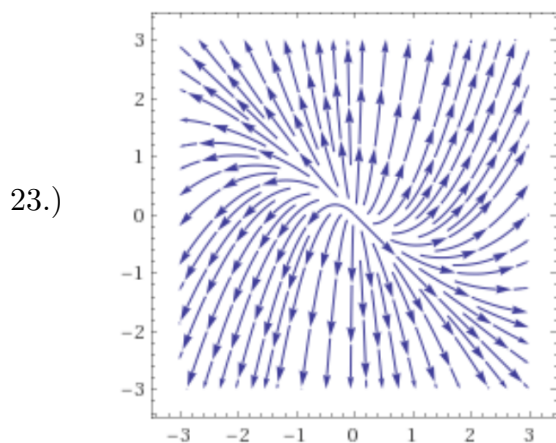
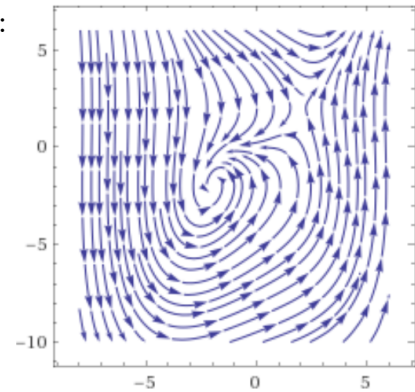
basin of attraction: $x_1 < 0$.



$(x_1(t), x_2(t)) = (2, 2)$ is an unstable saddle.

$(x_1(t), x_2(t)) = (-2, -2)$ is an asympt. stable spiral.

basin of attraction:



$(x_1(t), x_2(t)) = (0, 0)$ is an unstable node.

No basin of attraction: $x_1 < 0$.