**Summary of sections 3.1, 3, 4:** Solve linear homogeneous 2nd order DE with constant coefficients.

Solve $ay'' + by' + cy = 0$. Educated guess $y = e^{rt}$, then

$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $r_1 \neq r_2$, then $b^2 - 4ac \neq 0$. Hence a general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$

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If $b^2 - 4ac > 0$, general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$.

---

If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula.

general solution is $y = c_1e^{dt}cos(nt) + c_2e^{dt}sin(nt)$ where $r = d \pm in$

---

If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: $te^{r_1t}$

Hence general solution is $y = c_1e^{r_1t} + c_2te^{r_1t}$.

---

Initial value problem: use $y(t_0) = y_0$, $y'(t_0) = y_0'$ to solve for $c_1, c_2$ to find unique solution.

**Examples:**

Ex 1: Solve $y'' - 3y' - 4y = 0,\ \ y(0) = 1,\ y'(0) = 2.$

If $y = e^{rt}$, then $y' = re^{rt}$ and $y'' = r^2e^{rt}$.

$r^2e^{rt} - 3re^{rt} - 4e^{rt} = 0$

$r^2 - 3r - 4 = 0$ implies $(r-4)(r+1) = 0$. Thus $r = 4, -1$

Hence general solution is $y = c_1e^{4t} + c_2e^{-t}$

Solution to IVP:
Need to solve for 2 unknowns, $c_1$ & $c_2$; thus need 2 eqns:

$y = c_1e^{4t} + c_2e^{-t}$, $\quad y(0) = 1$ implies $1 = c_1 + c_2$

$y' = 4c_1e^{4t} - c_2e^{-t}$, $\quad y'(0) = 2$ implies $2 = 4c_1 - c_2$

Thus $3 = 5c_1$ & hence $c_1 = \frac{3}{5}$ and $c_2 = 1 - c_1 = 1 - \frac{3}{5} = \frac{2}{5}$

Thus IVP soln: $y = \frac{3}{5}e^{4t} + \frac{2}{5}e^{-t}$

---

Ex 2: Solve $y'' - 3y' + 4y = 0.$

$y = e^{rt}$ implies $r^2 - 3r + 4 = 0$ and hence

$$r = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(4)}}{2} = \frac{3}{2} \pm \frac{\sqrt{9-16}}{2} = \frac{3}{2} \pm i\frac{\sqrt{7}}{2}$$

Hence general sol'n is $y = c_1e^{\frac{3}{2}t}cos(\frac{\sqrt{7}}{2}t) + c_2e^{\frac{3}{2}t}sin(\frac{\sqrt{7}}{2}t)$

---

Ex 3: $y'' - 6y' + 9y = 0$ implies $r^2 - 6r + 9 = (r-3)^2 = 0$
Repeated root, $r = 3$ implies

general solution is $y = c_1e^{3t} + c_2te^{3t}$

So why did we guess $y = e^{rt}$?

Goal: Solve linear homogeneous 2nd order DE with constant coefficients,

$$ay'' + by' + cy = 0 \text{ where } a, b, c \text{ are constants}$$

Standard mathematical technique: make up simpler problems and see if you can generalize to the problem of interest.

Ex: linear homogeneous 1rst order DE: $y' + 2y = 0$

integrating factor $u(t) = e^{\int 2dt} = e^{2t}$

$y'e^{2t} + 2e^{2t}y = 0$

$(e^{2t}y)' = 0$. Thus $\int (e^{2t}y)'dt = \int 0dt$. Hence $e^{2t}y = C$

So $y = Ce^{-2t}$.

Thus exponential function could also be a solution to a linear homogeneous 2nd order DE

Ex: Simple linear homog 2nd order DE $y'' + 2y' = 0$.

Let $v = y'$, then $v' = y''$

$y'' + 2y' = 0$ implies $v' + 2v = 0$ implies $v = e^{2t}$.

Thus $v = y' = \frac{dy}{dt} = Ce^{-2t}$. Hence $dy = Ce^{-2t}dt$ and

$$y = c_1e^{-2t} + c_2.$$

$$y = c_1 e^{-2t} + c_2.$$

Note 2 integrations give us 2 constants.

Note also that the general solution is a linear combination of two solutions:

Let $c_1 = 1,\ c_2 = 0$, then we see, $y(t) = e^{-2t}$ is a solution.

Let $c_1 = 0,\ c_2 = 1$, then we see, $y(t) = 1$ is a solution.

The general solution is a linear combination of two solutions:

$$y = c_1 e^{-2t} + c_2(1).$$

---

Recall: you have seen this before:

Solve linear homogeneous matrix equation $A\mathbf{y} = \mathbf{0}$.

The general solution is a linear combination of linearly independent vectors that span the solution space:

$$\mathbf{y} = c_1\mathbf{v_1} + ...c_n\mathbf{v_n}$$

---

FYI: You could see this again:

Math 4050: Solve homogeneous linear recurrance relation $x_n - x_{n-1} - x_{n-2} = 0$ where $x_1 = 1$ and $x_2 = 1$.

Fibonacci sequence: $x_n = x_{n-1} + x_{n-2}$

1, 1, 2, 3, 5, 8, 13, 21, ...

Note $x_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$

Proof: $x_n = x_{n-1} + x_{n-2}$ implies $x_n - x_{n-1} - x_{n-2} = 0$

Suppose $x_n = r^n$. Then $x_{n-1} = r^{n-1}$ and $x_{n-2} = r^{n-2}$

Then $0 = x_n - x_{n-1} - x_{n-2} = r^n - r^{n-1} - r^{n-2}$

Thus $r^{n-2}(r^2 - r - 1) = 0$.

Thus either $r = 0$ or $r = \frac{1 \pm \sqrt{1-4(1)(-1)}}{2} = \frac{1\pm\sqrt{5}}{2}$

Thus $x_n = 0, \quad x_n = \left(\frac{1+\sqrt{5}}{2}\right)^n$ and $\quad f_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$

are 3 different sequences that satisfy the

homog linear recurrence relation: $x_n - x_{n-1} - x_{n-2} = 0$.

Hence $x_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$ also satisfies this

homogeneous linear recurrence relation.

Suppose the initial conditions are $x_1 = 1$ and $x_2 = 1$

Then for $n = 1$: $x_1 = 1$ implies $c_1 + c_2 = 1$

For $n = 2$: $x_2 = 1$ implies $c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$

We can solve this for $c_1$ and $c_2$ to determine that

$$x_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$$

## Existence and Uniqueness for LINEAR DEs.

Homogeneous:

$$y^{(n)} + p_1(t)y^{(n-1)} + ...p_{n-1}(t)y' + p_n(t)y = 0$$

Non-homogeneous: $g(t) \neq 0$

$$y^{(n)} + p_1(t)y^{(n-1)} + ...p_{n-1}(t)y' + p_n(t)y = g(t)$$

## 1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a, b) \to R$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the

$$\text{IVP: } y' + p(t)y = g(t), \quad y(t_0) = y_0$$

Thm: If $y = \phi_1(t)$ is a solution to homogeneous equation, $y' + p(t)y = 0$, then $y = c\phi_1(t)$ is the general solution to this equation.

If in addition $y = \psi(t)$ is a solution to non-homogeneous equation, $y' + p(t)y = g(t)$, then $y = c\phi_1(t) + \psi(t)$ is the general solution to this equation.

Partial proof: $y = \phi_1(t)$ is a solution to $y' + p(t)y = 0$ implies

Thus $y = c\phi_1(t)$ is a solution to $y' + p(t)y = 0$ since

$y = \psi(t)$ is a solution to $y' + p(t)y = g(t)$ implies

Thus $y = c\phi_1(t) + \psi(t)$ is a solution to $y' + p(t)y = g(t)$ since

## 2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a, b) \to R$, $q : (a, b) \to R$, and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$\begin{gathered} y'' + p(t)y' + q(t)y = g(t), \\ y(t_0) = y_0, \\ y'(t_0) = y_0' \end{gathered}$$

Thm 3.2.2: If $\phi_1$ and $\phi_2$ are two solutions to a <u>homogeneous</u> linear differential equation, then $c_1\phi_1 + c_2\phi_2$ is also a solution to this linear differential equation.

Proof of thm 3.2.2:

Since $y(t) = \phi_i(t)$ is a solution to the linear homogeneous differential equation $y'' + py' + qy = 0$ where $p$ and $q$ are functions of $t$ (note this includes the case with constant coefficients), then

Claim: $y(t) = c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution to $y'' + py' + qy = 0$

Pf of claim:

## Second order differential equation:

Linear equation with constant coefficients:

If the second order differential equation is

$$ay'' + by' + cy = 0,$$

$$\text{then } y = e^{rt} \text{ is a solution}$$

Need to have two independent solutions.

Solve the following IVPs:

1.) $y'' - 6y' + 9y = 0$ $\quad y(0) = 1,\ y'(0) = 2$

2.) $4y'' - y' + 2y = 0$ $\quad y(0) = 3,\ y'(0) = 4$

3.) $4y'' + 4y' + y = 0$ $\quad y(0) = 6,\ y'(0) = 7$

4.) $2y'' - 2y = 0$ $\quad y(0) = 5,\ y'(0) = 9$

**Summary of sections 3.1, 3, 4:** Solve linear homogeneous 2nd order DE with constant coefficients.

Solve $ay'' + by' + cy = 0$. Educated guess $y = e^{rt}$, then

$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $r_1 \neq r_2$, then $b^2 - 4ac \neq 0$. Hence a general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$

---

If $b^2 - 4ac > 0$, general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$.

---

If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula.

general solution is $y = c_1e^{dt}cos(nt) + c_2e^{dt}sin(nt)$ where $r = d \pm in$

---

If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: $te^{r_1t}$

Hence general solution is $y = c_1e^{r_1t} + c_2te^{r_1t}$.

---

Initial value problem: use $y(t_0) = y_0$, $y'(t_0) = y_0'$ to solve for $c_1, c_2$ to find unique solution.

Derivation of general solutions:

If $b^2 - 4ac > 0$ we guessed $e^{rt}$ is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

Section 3.3: If $b^2 - 4ac < 0$, :

Changed format of $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula:

$$e^{it} = cos(t) + isin(t)$$

Hence $e^{(d+in)t} = e^{dt}e^{int} = e^{dt}[cos(nt) + isin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$= c_1 e^{dt}[cos(nt) + isin(nt)] + c_2 e^{dt}[cos(-nt) + isin(-nt)]$$

$$= c_1 e^{dt}cos(nt) + ic_1 e^{dt}sin(nt) + c_2 e^{dt}cos(nt) - ic_2 e^{dt}sin(nt)$$

$$=(c_1 + c_2)e^{dt}cos(nt) + i(c_1 - c_2)e^{dt}sin(nt)$$

$$= k_1 e^{dt}cos(nt) + k_2 e^{dt}sin(nt)$$

Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$.
Hence one solution is $y = e^{r_1 t}$ Need second solution.

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.

How about $y = v(t)e^{rt}$?

$$y' = v'(t)e^{rt} + v(t)re^{rt}$$

$$y'' = v''(t)e^{rt} + v'(t)re^{rt} + v'(t)re^{rt} + v(t)r^2e^{rt}$$

$$= v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2e^{rt}$$

$$ay'' + by' + cy = 0$$

$$a(v''e^{rt} + 2v're^{rt} + vr^2e^{rt}) + b(v'e^{rt} + vre^{rt}) + cve^{rt} = 0$$

$$a(v''(t) + 2v'(t)r + v(t)r^2) + b(v'(t) + v(t)r) + cv(t) = 0$$

$$av''(t) + 2av'(t)r + av(t)r^2 + bv'(t) + bv(t)r + cv(t) = 0$$

$$av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0$$

$$av''(t) + (2a(\tfrac{-b}{2a}) + b)v'(t) + 0 = 0$$

since $ar^2 + br + c = 0$ and $r = \frac{-b}{2a}$

$av''(t) + (-b + b)v'(t) = 0.$ Thus $av''(t) = 0.$

Hence $v''(t) = 0$ and $v'(t) = k_1$ and $v(t) = k_1 t + k_2$

Hence $v(t)e^{r_1 t} = (k_1 t + k_2)e^{r_1 t}$ is a soln

Thus $te^{r_1 t}$ is a nice second solution.

Hence general solution is $y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$

Solve: $y'' + y = 0$, $y(0) = -1$, $y'(0) = -3$

$r^2 + 1 = 0$ implies $r^2 = -1$. Thus $r = \pm i$.

Since $r = 0 \pm 1i$, $y = k_1 cos(t) + k_2 sin(t)$.

Then $y' = -k_1 sin(t) + k_2 cos(t)$

$y(0) = -1$: $-1 = k_1 cos(0) + k_2 sin(0)$ implies $-1 = k_1$

$y'(0) = -3$: $-3 = -k_1 sin(0) + k_2 cos(0)$ implies $-3 = k_2$

Thus IVP solution: $y = -cos(t) - 3sin(t)$

---

**When does the following IVP have unique sol'n:**

IVP: $ay'' + by' + cy = 0$, $y(t_0) = y_0$, $y'(t_0) = y_1$.

Suppose $y = c_1\phi_1(t) + c_2\phi_2(t)$ is a solution to $ay'' + by' + cy = 0$. Then $y' = c_1\phi_1'(t) + c_2\phi_2'(t)$

$y(t_0) = y_0$: $y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0)$

$y'(t_0) = y_1$: $y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0)$

To find IVP solution, need to solve above system of two equations for the unknowns $c_1$ and $c_2$.

Note the IVP has a unique solution if and only if the above system of two equations has a unique solution for $c_1$ and $c_2$.

Note that in these equations $c_1$ and $c_2$ are the unknowns and $y_0, \phi_1(t_0), \phi_2(t_0)$, $y_1, \phi_1'(t_0), \phi_2'(t_0)$ are the constants. We can translate this linear system of equations into matrix form:

$$\begin{array}{c} c_1\phi_1(t_0) + c_2\phi_2(t_0) = y_0 \\ c_1\phi_1'(t_0) + c_2\phi_2'(t_0) = y_1 \end{array} \Rightarrow \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

Note this equation has a unique solution if and only if

$$det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \phi_1\phi_2' - \phi_1'\phi_2 \neq 0$$

Definition: The Wronskian of two differential functions, $\phi_1$ and $\phi_2$ is

$$W(\phi_1, \phi_2) = \phi_1\phi_2' - \phi_1'\phi_2 = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}$$

Examples:

1.) $\mathrm{W}(cos(t),\, sin(t)) = \begin{vmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{vmatrix}$

$$= cos^2(t) + sin^2(t) = 1 > 0.$$

2.) $\mathrm{W}(e^{dt}cos(nt),\, e^{dt}sin(nt)) =$

$$\begin{vmatrix} e^{dt}cos(nt) & e^{dt}sin(nt) \\ de^{dt}cos(nt) - ne^{dt}sin(nt) & de^{dt}sin(nt) + ne^{dt}cos(nt) \end{vmatrix}$$

$$=e^{dt}cos(nt)(de^{dt}sin(nt)+ne^{dt}cos(nt))-e^{dt}sin(nt)(de^{dt}cos(nt)-ne^{dt}sin(n$$

$$=e^{2dt}[cos(nt)(dsin(nt)+ncos(nt))-sin(nt)(dcos(nt)-nsin(nt))]$$

$$=e^{2dt}[dcos(nt)sin(nt)+ncos^2(nt)-dsin(nt)cos(nt)+nsin^2(nt)])$$

$$= e^{2dt}[ncos^2(nt) + nsin^2(nt)]$$

$$= ne^{2dt}[cos^2(nt) + sin^2(nt)] \;=\; ne^{2dt} > 0 \text{ for all } t.$$

Definition: The Wronskian of two differential functions, $f$ and $g$ is

$$W(f, g) = fg' - f'g = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

Thm 3.2.3: Suppose that
$\phi_1$ and $\phi_2$ are two solutions to $y'' + p(t)y' + q(t)y = 0$.
If $W(\phi_1, \phi_2)(t_0) = \phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0) \neq 0$, then
there is a unique choice of constants $c_1$ and $c_2$ such that $c_1\phi_1 + c_2\phi_2$ satisfies this homogeneous linear differential equation and initial conditions, $y(t_0) = y_0$, $y'(t_0) = y_0'$.

Thm 3.2.4: Given the hypothesis of thm 3.2.1, suppose that $\phi_1$ and $\phi_2$ are two solutions to

$$y'' + p(t)y' + q(t)y = 0.$$

If $W(\phi_1, \phi_2)(t_0) \neq 0$, for some $t_0 \in (a, b)$, then any solution to this homogeneous linear differential equation can be written as $y = c_1\phi_1 + c_2\phi_2$ for some $c_1$ and $c_2$.

Defn If $\phi_1$ and $\phi_2$ satisfy the conditions in thm 3.2.4, then $\phi_1$ and $\phi_2$ form a fundamental set of solutions to $y'' + p(t)y' + q(t)y = 0$.

Thm 3.2.5: Given any second order homogeneous linear differential equation, there exist a pair of functions which form a fundamental set of solutions.

# 3.3: Linear Independence and the Wronskian

Defn: $f$ and $g$ are linearly dependent if there exists constants $c_1, c_2$ such that $c_1 \neq 0$ or $c_2 \neq 0$ and

$$c_1 f(t) + c_2 g(t) = 0 \text{ for all } t \in (a, b)$$

Thm 3.3.1: If $f : (a, b) \to R$ and $g(a, b) \to R$ are differentiable functions on (a, b) and if $W(f, g)(t_0) \neq 0$ for some $t_0 \in (a, b)$, then $f$ and $g$ are linearly independent on $(a, b)$. Moreover, if $f$ and $g$ are linearly dependent on $(a, b)$, then $W(f, g)(t) = 0$ for all $t \in (a, b)$

If $c_1 f(t) + c_2 g(t) = 0$ for all $t$, then $c_1 f'(t) + c_2 g'(t) = 0$

Solve the following linear system of equations for $c_1, c_2$

$$\begin{aligned} c_1 f(t_0) + c_2 g(t_0) &= 0 \\ c_1 f'(t_0) + c_2 g'(t_0) &= 0 \end{aligned}$$

$$\begin{bmatrix} f(t_0) & g(t_0) \\ f'(t_0) & g'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$