Calculus pre-requisites you must know.

Derivative = slope of tangent line = rate.

Integral = area between curve and x-axis (where area can be negative).

The Fundamental Theorem of Calculus: Suppose $f$ continuous on $[a, b]$.

1.) If $G(x) = \int_a^x f(t) \, dt$, then $G'(x) = f(x)$.
   I.e., $\frac{d}{dx} \left[ \int_a^x f(t) \, dt \right] = f(x)$.

2.) $\int_a^b f(t) \, dt = F(b) - F(a)$ where $F$ is any antiderivative of $f$, that is $F' = f$.

Integration Pre-requisites:

* Integration by substitution

* Integration by parts

* Integration by partial fractions

Note: Partial fractions are also used in ch 6 for a different application.
Suppose $f$ is cont. on $(a, b)$ and the point $t_0 \in (a, b)$,

Solve IVP: \[ \frac{dy}{dt} = f(t), \quad y(t_0) = y_0 \]

\[ dy = f(t)dt \]

\[ \int dy = \int f(t)dt \]

\[ y = F(t) + C \] where $F$ is any anti-derivative of $F$.

Initial Value Problem (IVP): $y(t_0) = y_0$

\[ y_0 = F(t_0) + C \] implies $C = y_0 - F(t_0)$

Hence unique solution (if domain connected) to IVP:

\[ y = F(t) + y_0 - F(t_0) \]
1.1: Examples of differentiable equation:

1.) $F = ma = m \frac{dv}{dt} = mg - \gamma v$

2.) Mouse population increases at a rate proportional to the current population:

More general model: $\frac{dp}{dt} = rp - k$

where $p(t) =$ mouse population at time $t$,
$r =$ growth rate or rate constant,
$k =$ predation rate = # mice killed per unit time.
3.) Continuous compounding \( \frac{dS}{dt} = rS + k \)

where \( S(t) \) = amount of money at time \( t \),
\( r \) = interest rate,
\( k \) = constant deposit rate

Direction field = slope field = graph of \( \frac{dy}{dt} \) in \( t, y \)-plane.

*** can use slope field to determine behavior of \( y \) including as \( t \rightarrow \pm \infty \).

*** Equilibrium Solution = constant solution

Most differential equations do not have an equilibrium solution.

Initial value: A chosen point \((t_0, y_0)\) through which a solution must pass. I.e., \((t_0, y_0)\) lies on the graph of the solution that satisfies this initial value.

Initial value problem (IVP): A differential equation where initial value is specified.
An initial value problem can have 0, 1, or multiple equilibrium solutions.

existence of a solution

uniqueness of solution

1.3:

ODE (ordinary differential equation): single independent variable

Ex: \( \frac{dy}{dt} = ay + b \)

PDE (partial differential equation): several independent variables

Ex: \( \frac{\partial x y}{\partial x} = \frac{\partial x y}{\partial y} \)

order of differential eq’n: order of highest derivative

example of order \( n \): \( y^{(n)} = f(t, y, ..., y^{(n-1)}) \)

linear vs Non-linear

Linear: \( a_0 y^{(n)} + ... + a_{n-1} y' + a_n y = g(t) \)

where \( a_i \)’s are functions of \( t \)
Note for this linear equation, the left hand side is a linear combination of the derivatives of $y$ (denoted by $y^{(k)}$, $k = 0, \ldots, n$) where the coefficient of $y^{(k)}$ is a function of $t$ (denoted $a_k(t)$).

Linear: $a_0(t)y^{(n)} + \ldots + a_{n-1}(t)y' + a_n(t)y = g(t)$

Determine if linear or non-linear:
Ex: $ty'' - t^3y' - 3y = \sin(t)$
Ex: $2y'' - 3y' - 3y^2 = 0$

Show that for some value of $r$, $y = e^{rt}$ is a soln to the $1^{st}$ order linear homogeneous equation $2y' - 6y = 0$.

To show something is a solution, plug it in:

$y = e^{rt}$ implies $y' = re^{rt}$. Plug into $2y' - 6y = 0$:

$2re^{rt} + 6e^{rt} = 0$ implies $2r - 6 = 0$ implies $r = 3$

Thus $y = e^{3t}$ is a solution to $2y' - 6y = 0$.

Show $y = Ce^{3t}$ is a solution to $2y' - 6y = 0$.

$2y' - 6y = 2(Ce^{3t})' - 6(Ce^{3t}) = 2C(e^{3t})' - 6C(e^{3t})$

$= C[2(e^{3t})' - 6(e^{3t})] = C(0) = 0$. 
If \( y(0) = 4 \), then \( 4 = Ce^{3(0)} \) implies \( C = 4 \).

Thus by existence and uniqueness thm, \( y = 4e^{3t} \) is the unique solution to IVP: \( 2y' + 6y = 0, \ y(0) = 4 \).

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CH 2: Solve \( \frac{dy}{dt} = f(t, y) \) for special cases:

2.2: Separation of variables: \( N(y)dy = P(t)dt \)

2.1: First order linear eqn: \( \frac{dy}{dt} + p(t)y = g(t) \)

Ex 1: \( t^2y' + 2ty = tsin(t) \)

Ex 2: \( y' = ay + b \)

Ex 3: \( y' + 3t^2y = t^2, \ y(0) = 0 \)

Note: can use either section 2.1 method (integrating factor) or 2.2 method (separation of variables) to solve ex 2 and 3.

Ex 1: \( t^2y' + 2ty = sin(t) \)  
(note, cannot use separation of variables).

\[
\begin{align*}
  t^2y' + 2ty & = sin(t) \\
  (t^2y)' & = sin(t) \quad \text{implies} \quad \int (t^2y)'dt = \int \sin(t)dt \\
  \int (t^2y)'dt & = -cos(t) + C \quad \text{implies} \quad y = -t^{-2}cos(t) + Ct^{-2}
\end{align*}
\]
Ex. 2: Solve \( \frac{dy}{dt} = ay + b \) by separating variables:

\[
\frac{dy}{ay+b} = dt \quad \Rightarrow \quad \int \frac{dy}{ay+b} = \int dt \quad \Rightarrow \quad \frac{ln|ay+b|}{a} = t + C
\]

\( ln|ay + b| = at + C \) \quad \text{implies} \quad \( e^{ln|ay+b|} = e^{at+C} \)

\( |ay + b| = e^C e^{at} \) \quad \text{implies} \quad \( ay + b = \pm(e^C e^{at}) \)

\( ay = C e^{at} - b \) \quad \text{implies} \quad \( y = C e^{at} - \frac{b}{a} \)

Gen ex: Solve \( y' + p(x)y = g(x) \)

Let \( F(x) \) be an anti-derivative of \( p(x) \). Thus \( p(x) = F'(x) \)

\[
e^{F(x)}y' + [p(x)e^{F(x)}]y = g(x)e^{F(x)}
\]

\[e^{F(x)}y' + [F'(x)e^{F(x)}]y = g(x)e^{F(x)}\]

\[\left[e^{F(x)}y\right]' = g(x)e^{F(x)}\]

\[e^{F(x)}y = \int g(x)e^{F(x)} \, dx\]

\[y = e^{-F(x)} \int g(x)e^{F(x)} \, dx\]
2.3: Modeling with differential equations.

Suppose salty water enters and leaves a tank at a rate of 2 liters/minute.

Suppose also that the salt concentration of the water entering the tank varies with respect to time according to $Q(t) \cdot tsin(t^2)$ g/liters where $Q(t) =$ amount of salt in tank in grams. (Note: this is not realistic).

If the tank contains 4 liters of water and initially contains 5g of salt, find a formula for the amount of salt in the tank after $t$ minutes.

Let $Q(t) =$ amount of salt in tank in grams.

Note $Q(0) = 5$ g

rate in = (2 liters/min)($Q(t) \cdot tsin(t^2)$ g/liters) = $2Qtsin(t^2)$ g/min

rate out = (2 liters/min)($\frac{Q(t)g}{4\text{liters}}$) = $\frac{Q}{2}$ g/min

$$\frac{dQ}{dt} = \text{rate in - rate out} = 2Qtsin(t^2) - \frac{Q}{2}$$

$$\frac{dQ}{dt} = Q(2tsin(t^2) - \frac{1}{2}), \quad Q(0) = 5$$
This is a first order linear ODE. It is also a separable ODE. Thus can use either 2.1 or 2.2 methods.

Using the easier 2.2:

\[
\int \frac{dQ}{Q} = \int (2tsin(t^2) - \frac{1}{2})dt = \int 2tsin(t^2)dt - \int \frac{1}{2}dt
\]

Let \( u = t^2, \ du = 2tdt \)

\[ln|Q| = \int sin(u)du - \frac{t}{2} = -cos(u) - \frac{t}{2} + C\]

\[= -cos(t^2) - \frac{t}{2} + C\]

\[|Q| = e^{-cos(t^2) - \frac{t}{2} + C} = e^Ce^{-cos(t^2) - \frac{t}{2}}\]

\[Q = Ce^{-cos(t^2) - \frac{t}{2}}\]

\[Q(0) = 5 : \ 5 = Ce^{-1-0} = Ce^{-1}. \text{ Thus } C = 5e\]

Thus \( Q(t) = 5e \cdot e^{-cos(t^2) - \frac{t}{2}} \)

Thus \( Q(t) = 5e^{-cos(t^2) - \frac{t}{2} + 1} \)

Long-term behaviour:

\[Q(t) = 5(e^{-cos(t^2)})(e^{-\frac{t}{2}})e\]

As \( t \to \infty, e^{-\frac{t}{2}} \to 0, \) while \( 5(e^{-cos(t^2)})e \) are finite.

Thus as \( t \to \infty, Q(t) \to 0. \)
Note IVP, $y' = y^{1/3}, y(x_0) = 0$ has an infinite number of solutions,

while IVP, $y' = y^{1/3}, y(x_0) = y_0$ where $y_0 \neq 0$ has a unique solution.

Initial Value Problem: $y(t_0) = y_0$

Use initial value to solve for C.
Section 2.4: Existence and Uniqueness.

In general, for \( y' = f(t, y) \), \( y(t_0) = y_0 \), solution may or may not exist and solution may or may not be unique.

Example Non-unique: \( y' = y^{\frac{1}{3}} \)

\( y = 0 \) is a solution to \( y' = y^{\frac{1}{3}} \) since \( y' = 0 = 0^{\frac{1}{3}} = y^{\frac{1}{3}} \)

Suppose \( y \neq 0 \). Then \( \frac{dy}{dx} = y^{\frac{1}{3}} \) implies \( y^{-\frac{1}{3}} dy = dx \)

\( \int y^{-\frac{1}{3}} dy = \int dx \) implies \( \frac{3}{2} y^{\frac{2}{3}} = x + C \)

\( y^{\frac{2}{3}} = \frac{2}{3} x + C \) implies \( y = \pm \sqrt[3]{\frac{2}{3} x + C} \)

Suppose \( y(3) = 0 \). Then \( 0 = \sqrt[3]{(2 + C)^3} \Rightarrow C = -2 \).

The IVP, \( y' = y^{\frac{1}{3}} \), \( y(3) = 0 \), has an infinite \# of sol’ns including: \( y = 0 \), \( y = \sqrt[3]{\frac{2}{3} x - 2} \), \( y = -\sqrt[3]{\frac{2}{3} x - 2} \)
Examples: No solution:

Ex 1: \( y' = y' + 1 \)

Ex 2: \((y')^2 = -1\)

Ex 3 (IVP): \( \frac{dy}{dx} = y(1 + \frac{1}{x}) \), \( y(0) = 1 \)

\( \int \frac{dy}{y} = \int (1 + \frac{1}{x}) dx \) \quad \text{implies} \quad \ln|y| = x + \ln|x| + C \\
|y| = e^{x+\ln|x|+C} = e^x e^{\ln|x|} e^C = C|x|e^x = Cxe^x \\
y = \pm Cxe^x \) \quad \text{implies} \quad y = Cxe^x \\
y(0) = 1: \quad 1 = C(0)e^0 = 0 \) \quad \text{implies} \\
IVP \( \frac{dy}{dx} = y(1 + \frac{1}{x}) \), \( y(0) = 1 \) has no solution.

http://www.wolframalpha.com

slope field: \( \{1, y(1+1/x)\} / sqrt(1+y^2(1+1/x)^2) \)
Special cases:

Suppose $f$ is cont. on $(a, b)$ and the point $t_0 \in (a, b)$, Solve IVP: \( \frac{dy}{dt} = f(t), \ y(t_0) = y_0 \)

\[
dy = f(t)dt
\]

\[
\int dy = \int f(t)dt
\]

\[
y = F(t) + C \text{ where } F \text{ is any anti-derivative of } F.
\]

Initial Value Problem (IVP): $y(t_0) = y_0$

\[
y_0 = F(t_0) + C \text{ implies } C = y_0 - F(t_0)
\]

Hence unique solution (if domain connected) to IVP:

\[
y = F(t) + y_0 - F(t_0)
\]

First order linear differential equation:

Thm 2.4.1: If $p$ and $g$ are continuous on $(a, b)$ and the point $t_0 \in (a, b)$, then there exists a unique function $y = \phi(t)$ defined on $(a, b)$ that satisfies the following initial value problem:

\[
y' + p(t)y = g(t), \ y(t_0) = y_0.
\]
More general case (but still need hypothesis)

Thm 2.4.2: Suppose the functions
\[ z = f(t, y) \text{ and } z = \frac{\partial f}{\partial y}(t, y) \]
are continuous on \((a, b) \times (c, d)\) and the point \((t_0, y_0) \in (a, b) \times (c, d)\),
then there exists an interval \((t_0 - h, t_0 + h) \subset (a, b)\)
such that there exists a unique function \(y = \phi(t)\)
defined on \((t_0 - h, t_0 + h)\) that satisfies the following initial value problem:

\[ y' = f(t, y), \quad y(t_0) = y_0. \]

If possible without solving, determine where the solution exists for the following initial value problems:

If not possible without solving, state where in the \(ty\)-plane, the hypothesis of theorem 2.4.2 is satisfied. In other words, use theorem 2.4.2 to determine where for some interval about \(t_0\), a solution to IVP, \(y' = f(t, y), \ y(t_0) = y_0\) exists and is unique.

Example 1: \(ty' - y = 1, \ y(t_0) = y_0\)
Example 2: \( y' = \ln\left|\frac{t}{y}\right|, \quad y(3) = 6 \)

Example 3: \( (t^2 - 1)y' - \frac{t^3y}{t-4} = \ln|t|, \quad y(3) = 6 \)

Section 2.4 example: \( \frac{dy}{dt} = \frac{1}{(1-t)(2-y)} \)

\( F(y, t) = \frac{1}{(1-t)(2-y)} \) is continuous for all \( t \neq 1, \ y \neq 2 \)

\[
\frac{\partial F}{\partial y} = \frac{\partial \left( \frac{1}{(1-t)(2-y)} \right)}{\partial y} = \frac{1}{(1-t)} \frac{\partial (2-y)^{-1}}{\partial y} = \frac{1}{(1-t)(2-y)^2}
\]

\( \frac{\partial F}{\partial y} \) is continuous for all \( t \neq 1, \ y \neq 2 \)

Thus the IVP \( \frac{dy}{dt} = \frac{1}{(1-t)(2-y)}, \ y(t_0) = y_0 \) has a unique solution if \( t_0 \neq 1, \ y_0 \neq 2 \).

Note that if \( y_0 = 2, \ \frac{dy}{dt} = \frac{1}{(1-t)(2-y)}, \ y(t_0) = 2 \) has two solutions if \( t_0 \neq 1 \) (and if we allow vertical slope in domain. Note normally our convention will be to NOT allow vertical slope in domain of solution).

Note that if \( t_0 = 1, \ \frac{dy}{dt} = \frac{1}{(1-t)(2-y)}, \ y(1) = y_0 \) has no solutions.
\[(1, 1/((1 - t)(2 - y)))/\sqrt{1 + 1/((1 - t)(2 - y))^2}\]

Solve via separation of variables: \[\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}\]

\[
\int (2 - y)dy = \int \frac{dt}{1-t}
\]

\[2y - \frac{y^2}{2} = -ln|1 - t| + C\]

\[y^2 - 4y - 2ln|1 - t| + C = 0\]

\[y = \frac{4\pm\sqrt{16+4(2ln|1-t|+C)}}{2} = 2\pm\sqrt{4 + 2ln|1 - t| + C}\]

\[y = 2 \pm \sqrt{2ln|1 - t| + C}\]

**Find domain:**

\[2ln|1 - t| + C \geq 0\] and \[t \neq 1\] and \[y \neq 2\]
NOTE: the convention in this class to choose the largest possible connected domain where tangent line to solution is never vertical.

\[2\ln|1 - t| \geq -C\] and \(t \neq 1\) and \(y \neq 2\) implies

\[\ln|1 - t| > -\frac{C}{2}\]

Note: we want to find domain for this \(C\) and thus this \(C\) can’t swallow constants).

\[|1 - t| > e^{-\frac{C}{2}}\] since \(e^x\) is an increasing function.

\[1 - t < -e^{-\frac{C}{2}}\] or \(1 - t > e^{-\frac{C}{2}}\)

\[-t < -e^{-\frac{C}{2}} - 1\] or \(-t > e^{-\frac{C}{2}} - 1\)

Domain: \[\begin{cases} 
    t > e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 1 \\
    t < -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 1.
\end{cases}\]

Note: Domain is much easier to determine when the ODE is linear.

---

Find \(C\) given \(y(t_0) = y_0\): \(y_0 = 2 \pm \sqrt{2\ln|1 - t_0| + C}\)

\[\pm(y_0 - 2) = \sqrt{2\ln|1 - t_0| + C}\]
\[(y_0 - 2)^2 - 2ln|1 - t_0| = C\]

\[y = 2 \pm \sqrt{2ln|1 - t| + C}\]

\[y = 2 \pm \sqrt{2ln|1 - t| + (y_0 - 2)^2 - 2ln|1 - t_0|}\]

\[y = 2 \pm \sqrt{(y_0 - 2)^2 + ln\left(\frac{(1-t)^2}{(1-t_0)^2}\right)}\]

**Domain:** \[
\begin{cases} 
t > e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 1 \\
t < -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 1. 
\end{cases}
\]

\[e^{-\frac{C}{2}} = e^{-\frac{(y_0 - 2)^2 - 2ln|1-t_0|}{2}} = |1 - t_0|e^{-\frac{(y_0 - 2)^2}{2}}\]

**Domain:** \[
\begin{cases} 
t > 1 + |1 - t_0|e^{-\frac{(y_0 - 2)^2}{2}} & \text{if } t_0 > 1 \\
t < 1 - |1 - t_0|e^{-\frac{(y_0 - 2)^2}{2}} & \text{if } t_0 < 1. 
\end{cases}
\]

2.4 #27b. Solve Bernoulli’s equation,

\[y' + p(t)y = g(t)y^n,\]

when \(n \neq 0, 1\) by changing it

\[y^{-n}y' + p(t)y^{1-n} = g(t)\]

when \(n \neq 0, 1\) by changing it to a linear equation by substituting \(v = y^{1-n}\)
Example: Solve \( ty' + 2t^{-2}y = 2t^{-2}y^5 \)
Section 2.5: Autonomous equations: \( y' = f(y) \)

Example: Exponential Growth/Decay
Example: population growth/radioactive decay

\[ y' = ry, \quad y(0) = y_0 \text{ implies } y = y_0 e^{rt} \]

\( r > 0 \quad r < 0 \)

Example: Logistic growth: \( y' = h(y)y \)

Example: \( y' = r(1 - \frac{y}{K})y \)

\( y \text{ vs } f(y) \quad \text{slope field:} \)

Equilibrium solutions:

As \( t \to \infty \), if \( y > 0 \), \( y \to \)

Solution: \( y = \frac{y_0 K}{y_0 + (K-y_0)e^{-rt}} \)
Section 2.5 Autonomous equations: $y' = f(y)$

If given either differential equation $y' = f(y)$  
OR direction field:

Find equilibrium solutions and determine if stable, unstable, semi-stable.

Understand what the above means.

Asymptotically stable:

Asymptotically unstable:

Asymptotically semi-stable: