To solve linear DE $ay'' + by' + cy = g_1 + g_2 + g_3$

Step 1: Solve homogeneous version: $ay'' + by' + cy = 0$ implies
$ar^2 + br + c = 0$ implies ... $y = c_1\phi_1 + c_2\phi_2$.

Step 2a: Find one non-homogeneous solution, $y = f_1$, to $ay'' + by' + cy = g_1$
Step 2b: Find one non-homogeneous solution, $y = f_2$, to $ay'' + by' + cy = g_2$
Step 2c: Find one non-homogeneous solution, $y = f_3$, to $ay'' + by' + cy = g_3$

Step 3: Combine all solutions to create the general solution to the non-homogeneous DE:

$$y = c_1\phi_1 + c_2\phi_2 + f_1 + f_2 + f_3$$

Last step: If IVP, plug in initial values to find the constants c_1 and c_2.

Guess a possible non-homog soln for the following DEs:

Note homogeneous solution to $\boxed{y'' + 2y' + y = 0}$ is
$y = c_1e^{-t} + c_2te^{-t}$ since $r^2 + 2r + 1 = (r+1)(r+1) = 0$

1.) $y'' + 2y' + y = 4e^{2t}$

Guess: $y = Ae^{2t}$

2.) $y'' + 2y' + y = 4e^t$

Guess: $y = Ae^{t}$

3.) $y'' + 2y' + y = 4e^{-t}$

Guess: $y = At^2e^{-t}$

Plug $y = Ate^{-t}$ into LHS, we will get 0 on RHS since homog soln

So need to multiply by t again

$y = Ae^{-t}$ is a homogeneous soln so if we plug in $y = Ae^{-t}$, we will get 0. Thus we don't get $4e^{-t}$, so guess is incorrect for non homog soln

If guess is wrong, multiply by t

$2(A) + (At+B) = 1t + 0 \Rightarrow A = 1,\ 2A+B = 0$ (unique soln exists)

4.) $y'' + 2y' + y = t$

Guess: $y = At + B$

But $2A + At = 1t \Rightarrow 2A = 0,\ A = 1$ } no soln due to constant term, so add constant term

5.) $y'' + 2y' + y = t + 1$

Guess: $y = At + B$

(needed constant term since constant term appears on LHS)

6.) $y'' + 2y' + y = 4sin(2t)$

need $\cos(2t)$ term since will appear on LHS

Guess: $y = A\sin(2t) + B\cos(2t)$

7.) $y'' + 2y' + y = [4sin(2t) + 5cos(2t)] = g_1$

Guess: $y = A\cos(2t) + B\sin(2t)$

8.) $y'' + 2y' + y = [4sin(2t)] + [5cos(3t)] = g_1 + g_2$

Guess for step 2a: $y = A\cos(2t) + B\sin(2t)$

Guess for step 2b: $y = A\cos(3t) + B\sin(3t)$

9.) $y'' + 2y' + y = [4sin(2t)] + [t + 1] = g_1 + g_2$

$g_1 \rightarrow$ Guess for step 2a: $y = A\cos(2t) + B\sin(2t)$

$g_2 \rightarrow$ Guess for step 2b: $y = At + B$

10.) $y'' + 2y' + y = 4tsin(2t)$

product

$y = (At + B)(A\cos(2t) + B\sin(2t))$

product

simpler than plus in since will get 4 LINEAR equations for solve 4 unknowns

Guess: $y = a\,t\cos(2t) + b\cos(2t) + ct\sin(2t) + d\sin(2t)$

a, b, c, d

11.) $y'' + y = 4sin(2t)$ ← don't need $+B\cos(2t)$ since no y' term

Guess: $y = A\sin(2t)$

(but $y = A\sin(2t) + B\cos(2t)$ also works

12.) $y'' + y = 4sin(t)$

Guess: $y = At\cdot\sin(t) + Bt\cos(t)$

Solve homog $\Rightarrow r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow$ homog soln $y = c_1\cos(t) + c_2\sin(2t)$

$y = A\sin t$ is a homog soln $\Rightarrow$ so multiply NON SIMPLIFIED by t

$y = B\cos t$ is also homog

3.5: Solving non-homogeneous linear DE using the undetermined coefficients method

1.) Step 1: Solve homogeneous version of DE.

2.) Step 2: Guess a non-homogeneous solution with undetermined coefficients. Plug into the non-homogeneous linear DE to solve for the undetermined coefficients.

3.) Combing general homogeneous solution with a non-homogeneous solution.

Starting guess: 2nd order $\Rightarrow$ $y = \underbrace{c_1\phi_1 + c_2\phi_2}_{\text{general homog}} + \underset{\text{a non-homog}}{\psi}$

If $ay'' + by' + cy = ke^{pt}$, guess $y = Ae^{pt}$

If $ay'' + by' + cy = ksin(pt) + jcos(pt)$, guess $y = Asin(pt) + Bcos(pt)$ ok to commute

If $ay'' + by' + cy =$ degree n polynomial,
guess $y =$ a degree n polynomial including all terms (with undetermined coefficients) including constant term.

If $ay'' + by' + cy =$ a sum, guess a sum (but usually solve separately).

If $ay'' + by' + cy =$ a product, guess a product.

Sometimes the above can be simplified:

If a term does not show up when you take the derivatives of y, you may be able to omit that term. E.g, $y'' + w^2 y = sin(pt)$ where $p \neq w$, then $y = Asin(pt)$ is a simpler guess that works.

no y' term, so no cosine term appears on LHS when plug in guess, so don't need cosine term in guess

If the above does not work

Try multiplying non-simplified guess by t.

Example: If guess is a homogeneous solution, then that will not be a non-homogeneous solution. Thus must guess something else. Multiplying non-simplified guess by t until no longer homogeneous works.

Example: If y term missing, and $g(t) =$ degree n polynomial, then will need to multiply by t so that when you plug in guess, you will have a degree n polynomial on both sides of equal sign.

Note: you are multiplying the **non-simplified guess** by t. When you take derivatives of y, you must use the **product** rule. Thus extra terms appear when you take the derivative and you will need the non-simplified guess to cancel out these terms.

or see if a term shows up on LHS not in RHS, then add this term to guess