$Q_1(t)$ = amount of salt in tank 1 at time t

Assume flow rates preserve tank volumes

$$\left(\frac{oz}{min}\right) \quad \frac{dQ_1}{dt} = \frac{V_1\ gal}{min} \cdot \frac{1\,oz}{gal}$$

$$- \frac{V_{12}\ gal}{min} \cdot \frac{Q_1\ oz}{Vol_1}$$

$$+ \frac{V_{21}\ gal}{min} \cdot \frac{Q_2\ oz}{Vol_2}$$

$$\frac{dQ_2}{dt} = \frac{V_2\ gal}{min} \cdot \frac{3\,oz}{gal}$$

$$+ \frac{V_{12}\ gal}{min} \cdot \frac{Q_1\ oz}{Vol_1}$$

$$- (V_{21} + V_{20}) \frac{gal}{min} \frac{Q_2}{Vol_2}$$

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want equilibrium so solve $Q_1' = 0$

$Q_2' = 0$

Transform the system of equations

$$x_1' = 4x_1 + x_2 \quad (1)$$
$$x_2' = 6x_1 + 3x_2 \quad (2)$$

to one 2nd order DE by eliminating x_2

(1) $x_1' = 4x_1 + x_2 \Rightarrow x_2 = x_1' - 4x_1$
$\Rightarrow x_2' = x_1'' - 4x_1'$

Plug into (2) $x_2' = 6x_1 + 3x_2$

$$x_1'' - 4x_1' = 6x_1 + 3(x_1' - 4x_1)$$
$$= 6x_1 + 3x_1' - 12x_1$$

$$x_1'' - 7x_1' + 6x_1 = 0$$

Solve: $r^2 - 7r + 6 = (r-6)(r-1) = 0$

$\Rightarrow x_1 = c_1 e^{6t} + c_2 e^{t}$

$x_2 = x_1' - 4x_1 = 6c_1 e^{6t} + c_2 e^{t} - 4(c_1 e^{6t} + c_2 e^{t})$

$x_2 = 2c_1 e^{6t} - 3c_2 e^{t}$

$$x_1 = c_1 e^{6t} + c_2 e^{t}$$

$$x_2 = 2c_1 e^{6t} - 3c_2 e^{t}$$

Can also solve IVP per chapter 3 methods

Ch 7 method

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} e^{6t} \\ 2e^{6t} \end{bmatrix} + c_2 \begin{bmatrix} e^{t} \\ -3e^{t} \end{bmatrix}$$

IVP $\quad x_1(t_0) = a \qquad x_2(t_0) = b$

$$\begin{bmatrix} a \\ b \end{bmatrix} = c_1 \begin{bmatrix} e^{6t_0} \\ 2e^{6t_0} \end{bmatrix} + c_2 \begin{bmatrix} 1e^{t_0} \\ -3e^{t_0} \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} e^{6t_0} & e^{t_0} \\ 2e^{6t_0} & -3e^{t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

unique solution $\Longleftrightarrow$

Wronskian

$$W\left(\begin{bmatrix} e^{6t} \\ 2e^{6t} \end{bmatrix}, \begin{bmatrix} e^{t} \\ -3e^{t} \end{bmatrix} \right) = \begin{vmatrix} e^{6t} & e^{t} \\ 2e^{6t} & -3e^{t} \end{vmatrix} = -3e^{7t} - 2e^{7t}$$

$$= -5e^{7t} \neq 0$$

$$W\left(\begin{bmatrix}5t^2\\10t\end{bmatrix}, \begin{bmatrix}e^t\\e^t\end{bmatrix}\right) = \begin{vmatrix}5t^2 & e^t\\10t & e^t\end{vmatrix}$$

$$= 5t^2e^t - 10te^t = te^t(5t - 10)$$

linear indep if $W \neq 0$

$$te^t(5t - 10) = 0 \Rightarrow t = 0, 2$$

$\Rightarrow$ lin indep in

$$(-\infty, 0) \cup (0, 2) \cup (2, \infty)$$

$$\vec{x}\,' = \begin{pmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{pmatrix}\vec{x}$$

has unique soln iff all p_{ij} are cont

possibly not unique soln for $t = 0, 2$

$\Rightarrow \exists\ p_{ij}$ disc at $t = 0$ & a p_{lk} disc at $t = 2$

$$\vec{X} = c_1 \begin{bmatrix} 5t^2 \\ 10t \end{bmatrix} + c_2 \begin{bmatrix} e^t \\ e^t \end{bmatrix} = \begin{bmatrix} 5c_1 t^2 + c_2 e^t \\ 10c_1 t + c_2 e^t \end{bmatrix}$$

is a soln to $x' = Px$

Find a possible P

plug in $\vec{X}$

$$\begin{bmatrix} 5c_1 t^2 + c_2 e^t \\ 10c_1 t + c_2 e^t \end{bmatrix}' = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 5c_1 t^2 + c_2 e^t \\ 10c_1 t + c_2 e^t \end{bmatrix}$$

$$\begin{bmatrix} 10c_1 t + c_2 e^t \\ 10c_1 + c_2 \end{bmatrix} = \begin{bmatrix} p_{11}(5c_1 t^2 + c_2 e^t) + p_{12}(10c_1 t + c_2 e^t) \\ p_{21}(5c_1 t^2 + c_2 e^t) + p_{22}(10c_1 t + c_2 e^t) \end{bmatrix}$$

1rst row: $10c_1 t + c_2 e^t = 5c_1 p_{11} t^2 + c_2 p_{11} e^t + p_{12} 10c_1 t + p_{12} c_2 e^t$

Must hold for all c_1 and c_2

$\left.\begin{matrix} c_1 = 1 \\ c_2 = 0 \end{matrix}\right\}$: $\quad 10t = 5p_{11}t^2 + p_{12} 10t$

$\quad 10 = 5p_{21}t^2 + p_{22} 10t$

$\left.\begin{matrix} c_2 = 0 \\ c_1 = 1 \end{matrix}\right\}$:

4 eqns ~~4~~ solve for 4 unknown fns $p_{11}, p_{12}, p_{21}, p_{22}$