Equilibrium sol'n = constant sol'n

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \text{critical pt}$$

Ch 7 and 9

Suppose an object moves in the 2D plane (the x_1, x_2 plane) so that it is at the point $(x_1(t), x_2(t))$ at time t. Suppose the object's velocity is given by

$$x_1'(t) = ax_1 + bx_2,$$
$$x_2'(t) = cx_1 + dx_2$$

Or in matrix form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

To solve, find eigenvalues and corresponding eigenvectors:

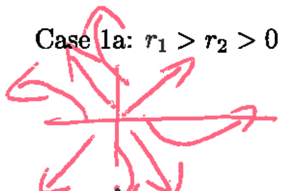
$$\begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = (a-r)(d-r) - bc = r^2 - (a+d)r + ad - bc = 0.$$

Thus $\boxed{r = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}}$

Case 1: $(a+d)^2 - 4(ad-bc) > 0$ Two real e. vectors

Hence the general solutions is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{r_1 t} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{r_2 t}$

Case 1a: $r_1 > r_2 > 0$



The critical pt $(0,0)$ is an unstable node = source

Case 1b: $r_1 < r_2 < 0$

The critical pt, $(0,0)$ is a asym stable node = sink

Case 1c: $r_2 < 0 < r_1$

The critical pt, $(0,0)$ is an unstable saddle

Case 2: $(a+d)^2 - 4(ad-bc) = 0$

Case 2i: Two independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{rt}$

Case 2ii: One independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} t + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] e^{rt}$

Case 2a: $r > 0$

Case 2b: $r < 0$

Case 3: $(a+d)^2 - 4(ad-bc) < 0$. I.e., $r = \lambda \pm i\mu$

Suppose the eigenvector corresponding to this eigenvalue is

$$\begin{pmatrix} v_1 + iw_1 \\ v_2 + iw_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + i \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Then general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 cos(\mu t) - w_1 sin(\mu t) \\ v_2 cos(\mu t) - w_2 sin(\mu t) \end{pmatrix} e^{\lambda t} + c_2 \begin{pmatrix} v_1 sin(\mu t) + w_1 cos(\mu t) \\ v_2 sin(\mu t) + w_2 cos(\mu t) \end{pmatrix} e^{\lambda t}$$

Case 3a: $\lambda > 0$

Case 3a: $\lambda < 0$

Case 3a: $\lambda = 0$

$mr'' = \frac{-GMm}{r^2}$

Let $v = r'$, then $v' = r''$

Thus we obtain system of non-linear equations:

$$r' = v$$
$$v' = \frac{-GM}{r^2}$$

Note $v' = \frac{-GM}{r^2}$ involves 3 variables: v, t, r

Eliminate t: $\quad v' = \frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = \frac{dv}{dr}v$

Thus $mv' = \frac{-GMm}{r^2}$ becomes $m\frac{dv}{dr}v = \frac{-GMm}{r^2}$

Separate variables: $\int mdvv = \int \frac{-GMm}{r^2}dr$

$$\frac{1}{2}mv^2 = \frac{GMm}{r} + E \text{ where } E \text{ is a constant.}$$

Thus we have derived the physics formula, conservation of energy:

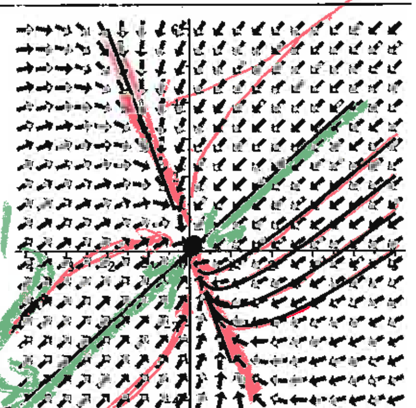
$$\frac{1}{2}mv^2 + \frac{-GMm}{r} = E$$

I.e., Kinetic Energy + Potential Energy = constant

slope $= -3/1$

$x' = -4x - y$
$y' = -3x \mp 2y$

$\vec{X}' = \begin{bmatrix} -4 & -1 \\ -3 & -2 \end{bmatrix} \vec{X}$



evector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

since slope = 1

$\begin{bmatrix} -4 & -1 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \end{bmatrix}$

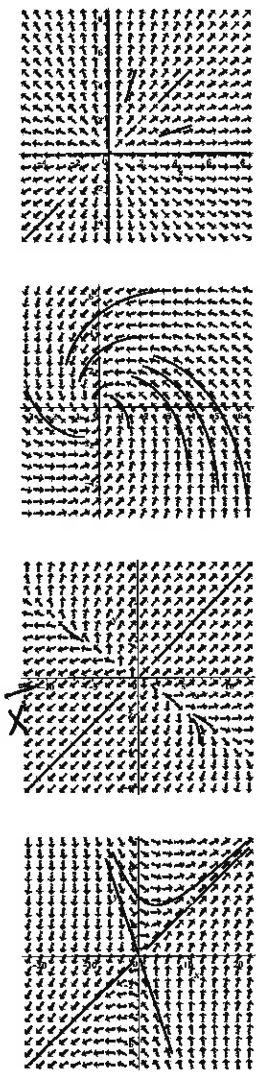
$r_1 = -5$ eigenvalues

$r_1, r_2 < 0$

$\begin{bmatrix} -4 & -1 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$r_2 = -1$

Suppose the following represent direction fields of linear systems of 1rst order differential equations in the phase plane. What can you say about solutions to these systems of equations.



Every point

are ellipses when the

(i)

maginary if and only if

(ii)

Eqs. (i) into the single

(iii)

(iv)

(a) Node if $q > 0$ and $\Delta \geq 0$; (b) Saddle point if $q < 0$;

(c) Spiral point if $p \neq 0$ and $\Delta < 0$; (d) Center if $p = 0$ and $q > 0$.

Hint: These conclusions can be reached by studying the eigenvalues r_1 and r_2. It may also be helpful to establish, and then to use, the relations $r_1 r_2 = q$ and $r_1 + r_2 = p$.

21. Continuing Problem 20, show that the critical point $(0, 0)$ is

(a) Asymptotically stable if $q > 0$ and $p < 0$;

(b) Stable if $q > 0$ and $p = 0$;

(c) Unstable if $q < 0$ or $p > 0$.

The results of Problems 20 and 21 are summarized visually in Figure 9.1.9.

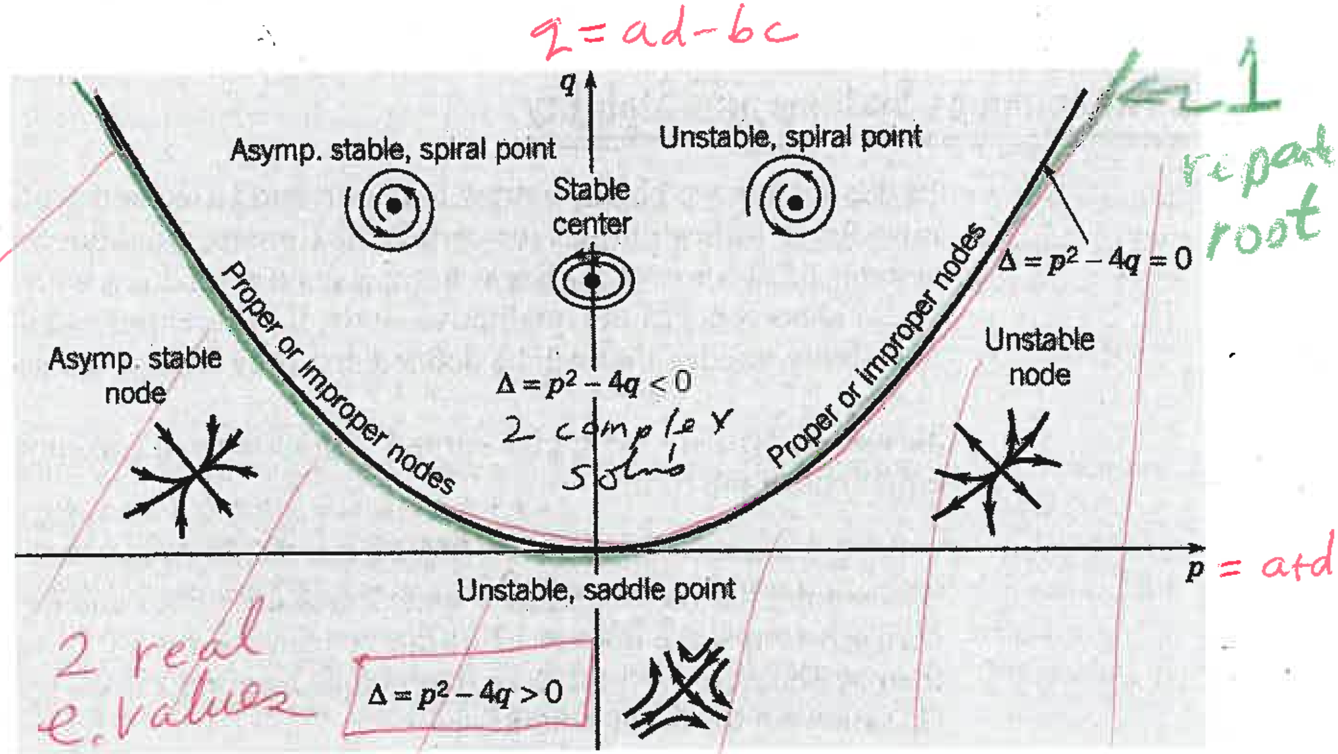


FIGURE 9.1.9 Stability diagram.

22. In this problem we illustrate how a 2×2 system with eigenvalues $\lambda \pm i\mu$ can be transformed into the system (11). Consider the system in Problem 12:

$$\mathbf{x}' = \begin{pmatrix} 2 & -2.5 \\ 1.8 & -1 \end{pmatrix} \mathbf{x} = \mathbf{A}\mathbf{x}. \qquad \text{(i)}$$

(a) Show that the eigenvalues of this system are $r_1 = 0.5 + 1.5i$ and $r_2 = 0.5 - 1.5i$.

(b) Show that the eigenvector corresponding to r_1 can be chosen as