Guess $y = |x|^r$

Solve $x^2y'' + \alpha xy' + \beta y = 0$. Let $y = x^r$,
$y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$ (case when $y = (-x)^r$ is similar).

$x^2x^{r-2}r(r-1) + \alpha x x^{r-1}r + \beta x^r = 0$

$x^r[r^2 - r + \alpha r + \beta] = 0$ for all x implies $r^2 + (\alpha - 1)r + \beta = 0$

Thus x^r is a solution iff $r = \frac{-(\alpha-1)\pm\sqrt{(\alpha-1)^2-4\beta}}{2}$

Case 1: Two real roots, r_1, r_2.

General solution is $y = c_1|x|^{r_1} + c_2|x|^{r_2}$

Case 2: Two complex roots, $r_i = \lambda \pm i\mu$:

Convert solution to form without complex numbers.

Note $|x|^{\pm i\mu} = e^{ln(|x|^{\pm i\mu})} = e^{(\pm i\mu)ln|x|} = e^{i(\pm \mu ln|x|)}$

$$= cos(\pm\mu ln|x|) + isin(\pm\mu ln|x|)$$
$$= cos(\mu ln|x|) \pm isin(\mu ln|x|)$$

General solution is $y = c_1|x|^{r_1} + c_2|x|^{r_2} = c_1|x|^{\lambda+i\mu} + c_2|x|^{\lambda-i\mu}$

$= |x|^\lambda(c_1|x|^{i\mu} + c_2|x|^{-i\mu})$

$= |x|^\lambda(c_1[cos(\mu ln|x|) + isin(\mu ln|x|)] + c_2[cos(\mu ln|x|) - isin(\mu ln|x|)])$

$= |x|^\lambda([c_1 + c_2]cos(\mu ln|x|) + i[c_1 - c_2]sin(\mu ln|x|))$

$= |x|^\lambda(k_1cos(\mu ln|x|) + k_2sin(\mu ln|x|))$

$= k_1|x|^\lambda cos(\mu ln|x|) + k_2|x|^\lambda sin(\mu ln|x|)$

Case 3: one repeated root, $r_1 = \frac{-(\alpha-1)}{2}$. (i.e., $\sqrt{(\alpha-1)^2 - 4\beta} = 0$): ▮

Thus $|x|^{r_1}$ is a solution. Find 2nd solution.

Method 1. Reduction of order: Suppose $y = u(x)|x|^{r_1}$ is a solution to $x^2y'' + \alpha xy' + \beta y = 0$. Plug in and determine $u(x)$

This is what we did in 3.4 to find 2nd sol'n when repeated root

Method 2: Let $L(y) = x^2y'' + \alpha xy' + \beta y$ where $y' = \frac{dy}{dx}$.

$L(|x|^r) = |x|^r(r - r_1)^2$

$\frac{\partial}{\partial r}[L(|x|^r)] = \frac{\partial}{\partial r}[|x|^r(r-r_1)^2] = (|x|^r)'(r-r_1)^2 + 2|x|^r(r-r_1) = 0$ if $r = r_1$.

Suppose x is constant with respect to r and all the partial derivatives are continuous. Then

$$\frac{\partial}{\partial r}[L(y)] = \frac{\partial}{\partial r}[x^2y'' + \alpha xy' + \beta y] = x^2\frac{\partial y''}{\partial r} + \alpha x\frac{\partial y'}{\partial r} + \beta\frac{\partial y}{\partial r}$$
$$= x^2\frac{\partial}{\partial r}[\frac{\partial^2 y}{\partial x^2}] + \alpha x\frac{\partial}{\partial r}[\frac{\partial y}{\partial x}] + \beta\frac{\partial y}{\partial r}$$
$$= x^2\frac{\partial^2}{\partial x^2}[\frac{\partial y}{\partial r}] + \alpha x\frac{\partial}{\partial x}[\frac{\partial y}{\partial r}] + \beta\frac{\partial y}{\partial r}$$
$$= L(\frac{\partial y}{\partial r}) \text{ for all } r$$

$L(\frac{\partial|x|^r}{\partial r}) = \frac{\partial}{\partial r}[L(|x|^r)] = 0$ for $r = r_1$.

$\frac{\partial|x|^r}{\partial r} = \frac{\partial e^{ln|x|^r}}{\partial r}\ \frac{\partial e^{rln|x|}}{\partial r} = (e^{rln|x|})ln|x| = |x|^r ln|x|$

Thus $|x|^{r_1}ln|x|$ is a solution.

Thus general solution is $y = c_1|x|^{r_1} + c_2|x|^{r_1}ln|x|$

since by the Wronskian, $|x|^{r_1}$ and $|x|^{r_1}ln|x|$ are linearly independent. Suppose $x > 0$ and $r_1 \neq 0$.

$$\begin{vmatrix} x^{r_1} & x^{r_1}ln|x| \\ r_1x^{r_1-1} & r_1x^{r_1-1}ln|x| + x^{r_1-1} \end{vmatrix}$$

$$= x^{r_1}(r_1x^{r_1-1}ln|x| + x^{r_1-1}) - x^{r_1}ln|x|r_1x^{r_1-1}$$

$$= x^{2r_1-1}[r_1ln|x| + 1 - ln|x|r_1] = x^{2r_1-1} \neq 0 \text{ for } x \neq 0$$

Other cases for Wronskian are similar.

Section 5.4 continued

Solve $x^2y'' - 2xy' = 0$ (*).

We could solve by letting $v = y'$, but we will instead use 5.4 methods

Note x is an ordinary point iff $x \neq 0$ $\quad (y'' - \frac{2}{x}y' = 0.)$
$x = 0$ is a singular point.

Note $x^2x^{r-2}r(r-1) - 2xx^{r-1}r = 0$ implies $r^2 - r - 2r = 0$ and recall $y = (-x)^r$ gives same equation for r as $y = x^r$.

Thus $y = |x|^r$ implies $r^2 + (\alpha - 1)r + \beta = r^2 - 3r + 0 = r(r-3) = 0$

Thus $r = 0, 3$. Thus $y = |x|^0 = 1$ and $y = |x|^3$ are solutions to (*)

Since (*) is a linear equation, the general solution is $y = c_1 + c_2|x|^3$.

Note an equivalent general solution is $y = k_1 + k_2x^3$.

Both forms are valid for all x.

When is a unique solution to the following initial value problem guaranteed?

$$x^2y'' - 2xy' = 0, \quad y(t_0) = y_0, \ \ y'(t_0) = y_1 \ (**)$$

$$y'' - \frac{2}{x}y' = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1$$

Since $\frac{2}{x}$ and the zero constant function are continuous on $(-\infty, 0) \cup (0, \infty)$,

(**) has a unique solution for $t_0 < 0$ and this solution exists on $(-\infty, 0)$.

(**) has a unique solution for $t_0 > 0$ and this solution exists on $(0, \infty)$.

There are an infinite number of solutions for $y(0) = a$, $y'(0) = 0$.

How is x^r defined:

If n is a positive integer: $x^n = x \cdot x \cdot \ldots \cdot x$

If m is a positive integer: If $f(x) = x^m$, then $f^{-1}(x) = x^{\frac{1}{m}}$ and $x^{\frac{n}{m}} = (x^n)^{\frac{1}{m}}$

Let $r \geq 0$. Let r_n be any sequence consisting of positive rational numbers such that $lim_{n \to \infty} r_n = r$. Then
$$x^r = lim_{n \to \infty} x^{r_n}.$$

See more advanced class for why the above is well-defined.

If $r < 0$, then $x^r = x^{-r}$.

If x is a real number, when is x^r a real number?

$x^n = x \cdot x \cdot \ldots \cdot x$ is a real number when n is a positive integer.

If $f(x) = x^n$, then the image of $f = \begin{cases} \text{real numbers} & n \text{ odd} \\ [0, \infty) & n \text{ even} \end{cases}$

Thus if $f^{-1}(x) = x^{\frac{1}{n}}$ is real-valued, then
the domain of f^{-1} is $\begin{cases} \text{real numbers} & n \text{ odd} \\ [0, \infty) & n \text{ even} \end{cases}$

In complex analysis, $\left(\frac{1+i\sqrt{3}}{2}\right)^3 = -1$, $(-1)^3 = -1$, $\left(\frac{1-i\sqrt{3}}{2}\right)^3 = -1$

Recall $\left(e^{\frac{i\pi}{3}}\right)^3 = (cos\frac{\pi}{3} + isin\frac{\pi}{3})^3 = -1$

Complex numbers are also roots of unity:

$$\left(e^{\frac{2i\pi}{3}}\right)^3 = 1 \quad \left(e^{\frac{-2i\pi}{3}}\right)^3 = 1, \quad (1)^3 = 1$$

Derivation of general solutions:

If $b^2 - 4ac > 0$ we guessed e^{rt} is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

Section 3.3: If $b^2 - 4ac < 0$, :

Changed format of $y = c_1e^{r_1t} + c_2e^{r_2t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula:

$$e^{it} = cos(t) + isin(t)$$

Hence $e^{(d+in)t} = e^{dt}e^{int} = e^{dt}[cos(nt) + isin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$$y = c_1e^{r_1t} + c_2e^{r_2t}$$
$$= c_1e^{dt}[cos(nt) + isin(nt)] + c_2e^{dt}[cos(-nt) + isin(-nt)]$$
$$= c_1e^{dt}cos(nt) + ic_1e^{dt}sin(nt) + c_2e^{dt}cos(nt) - ic_2e^{dt}sin(nt)$$
$$= (c_1 + c_2)e^{dt}cos(nt) + i(c_1 - c_2)e^{dt}sin(nt)$$
$$= k_1e^{dt}cos(nt) + k_2e^{dt}sin(nt)$$

Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$.
Hence one solution is $y = e^{r_1t}$ Need second solution.

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.

How about $y = \underline{v(t)e^{rt}}$?

$$y' = v'(t)e^{rt} + v(t)re^{rt}$$
$$y'' = v''(t)e^{rt} + v'(t)re^{rt} + v'(t)re^{rt} + v(t)r^2e^{rt}$$
$$= v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2e^{rt}$$

$$ay'' + by' + cy = 0$$

$$a(v''e^{rt} + 2v're^{rt} + vr^2e^{rt}) + b(v'e^{rt} + vre^{rt}) + cve^{rt} = 0$$

$$a(v''(t) + 2v'(t)r + v(t)r^2) + b(v'(t) + v(t)r) + cv(t) = 0$$

$$av''(t) + 2av'(t)r + av(t)r^2 + bv'(t) + bv(t)r + cv(t) = 0$$

$$av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0$$

$$av''(t) + (2a(\tfrac{-b}{2a}) + b)v'(t) + 0 = 0$$

since $ar^2 + br + c = 0$ and $r = \frac{-b}{2a}$

$av''(t) + (-b + b)v'(t) = 0$. Thus $av''(t) = 0$.

Hence $v''(t) = 0$ and $v'(t) = k_1$ and $v(t) = k_1t + k_2$

Hence $v(t)e^{r_1t} = (k_1t + k_2)e^{r_1t}$ is a soln

Thus te^{r_1t} is a nice second solution.

Hence general solution is $y = c_1e^{r_1t} + c_2te^{r_1t}$

Ch 5

Paul's Online Math Notes
Differential Equations (Notes) / Series Solutions to DE's / Euler Equations [Notes]
Differential Equations - Notes

Euler Equations

In this section we want to look for solutions to

$$\boxed{ax^2y'' + bxy' + cy = 0} \tag{1}$$

around $x_0 = 0$. These type of differential equations are called **Euler Equations.**

Recall from the previous section that a point is an ordinary point if the quotients,

$$\frac{bx}{ax^2} = \frac{b}{ax} \qquad \text{and} \qquad \frac{c}{ax^2}$$

have Taylor series around $x_0 = 0$. However, because of the x in the denominator neither of these will have a Taylor series around $x_0 = 0$ and so $x_0 = 0$ is a singular point. So, the method from the previous section won't work since it required an ordinary point.

However, it is possible to get solutions to this differential equation that aren't series solutions. Let's start off by assuming that $x>0$ (the reason for this will be apparent after we work the first example) and that all solutions are of the form,

$$y(x) = x^r \tag{2}$$

Now plug this into the differential equation to get,

$$ax^2(r)(r-1)x^{r-2} + bx(r)x^{r-1} + cx^r = 0$$
$$ar(r-1)x^r + b(r)x^r + cx^r = 0$$
$$(ar(r-1) + b(r) + c)x^r = 0$$

Now, we assumed that $x>0$ and so this will only be zero if,

$$ar(r-1) + b(r) + c = 0 \tag{3}$$

So solutions will be of the form (2) provided r is a solution to (3). This equation is a quadratic in r and so we will have three cases to look at : Real, Distinct Roots, Double Roots, and Complex Roots.

Real, Distinct Roots
There really isn't a whole lot to do in this case. We'll get two solutions that will form a fundamental set of solutions (we'll leave it to you to check this) and so our general solution will be,

$$y(x) = c_1 x^{r_1} + c_2 x^{r_2}$$

Example 1 Solve the following IVP

$$2x^2y'' + 3xy' - 15y = 0, \qquad y(1) = 0 \quad y'(1) = 1$$

initial value $x_0 = 1 > 0$ so can drop | |

Solution
We first need to find the roots to (3).

$$2r(r-1) + 3r - 15 = 0$$
$$2r^2 + r - 15 = (2r-5)(r+3) = 0 \qquad \Rightarrow \qquad r_1 = \frac{5}{2},\ r_2 = -3$$

The general solution is then,

$$y(x) = c_1 x^{\frac{5}{2}} + c_2 x^{-3}$$

To find the constants we differentiate and plug in the initial conditions as we did back in the second order differential equations chapter.

$$y'(x) = \frac{5}{2}c_1 x^{\frac{3}{2}} - 3c_2 x^{-4}$$

$2x^2y'' + 3xy' - 15y = 0$
$y(-1) = 0, \ y'(-1) = 1$
General soln: $y = c_1|x|^{5/2} + c_2 x^{-3}$

$$\left.\begin{aligned} 0 &= y(1) = c_1 + c_2 \\ 1 &= y'(1) = \frac{5}{2}c_1 - 3c_2 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = \frac{2}{11},\ c_2 = -\frac{2}{11}$$

The actual solution is then,

$$y(x) = \frac{2}{11}x^{\frac{5}{2}} - \frac{2}{11}x^{-3}$$

With the solution to this example we can now see why we required $x>0$. The second term would have division by zero if we allowed $x=0$ and the first term would give us square roots of negative numbers if we allowed $x<0$.

Double Roots

This case will lead to the same problem that we've had every other time we've run into double roots (or double eigenvalues). We only get a single solution and will need a second solution. In this case it can be shown that the second solution will be,

$$y_2(x) = x^r \ln x$$

and so the general solution in this case is,

$$y(x) = c_1 x^r + c_2 x^r \ln x = x^r\left(c_1 + c_2 \ln x\right)$$

We can again see a reason for requiring $x>0$. If we didn't we'd have all sorts of problems with that logarithm.

Example 2 Find the general solution to the following differential equation.

$$x^2 y'' - 7xy' + 16y = 0$$

$X \neq 0$ ordinary pts

Solution

First the roots of (3).

$$\begin{aligned} r(r-1) - 7r + 16 &= 0 \\ r^2 - 8r + 16 &= 0 \\ (r-4)^2 &= 0 \qquad \Rightarrow \qquad r = 4 \end{aligned}$$

So the general solution is then,

$$y(x) = c_1 x^4 + c_2 x^4 \ln|x|$$

Complex Roots

In this case we'll be assuming that our roots are of the form,

$$r_{1,2} = \lambda \pm \mu i$$

If we take the first root we'll get the following solution.

$$x^{\lambda + \mu i}$$

This is a problem since we don't want complex solutions, we only want real solutions. We can eliminate this by recalling that,

$$x^r = \mathbf{e}^{\ln x^r} = \mathbf{e}^{r \ln x}$$

Plugging the root into this gives,

$$\begin{aligned} x^{\lambda + \mu i} &= \mathbf{e}^{(\lambda + \mu i)\ln x} \\ &= \mathbf{e}^{\lambda \ln x}\mathbf{e}^{\mu i \ln x} \\ &= \mathbf{e}^{\ln x^\lambda}\left(\cos(\mu \ln x) + i \sin(\mu \ln x)\right) \\ &= x^\lambda \cos(\mu \ln x) + i x^\lambda \sin(\mu \ln x) \end{aligned}$$

Note that we had to use Euler formula as well to get to the final step. Now, as we've done every other time we've seen solutions like this we can take the real part and the imaginary part and use those for our two solutions.

So, in the case of complex roots the general solution will be,

$$y(x) = c_1 x^\lambda \cos(\mu \ln x) + c_2 x^\lambda \sin(\mu \ln x) = x^\lambda \left(c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x)\right)$$

Once again we can see why we needed to require $x>0$.

Example 3 Find the solution to the following differential equation.

$$x^2 y'' + 3xy' + 4y = 0$$

Solution

Get the roots to (3) first as always.

$$r(r-1) + 3r + 4 = 0$$

$$r^2 + 2r + 4 = 0 \qquad \Rightarrow \qquad r_{1,2} = -1 \pm \sqrt{3}\, i$$

The general solution is then,

$$y(x) = c_1 x^{-1} \cos\left(\sqrt{3} \ln x\right) + c_2 x^{-1} \sin\left(\sqrt{3} \ln x\right)$$

We should now talk about how to deal with $x<0$ since that is a possibility on occasion. To deal with this we need to use the variable transformation,

$$\eta = -x$$

In this case since $x<0$ we will get $\eta>0$. Now, define,

$$u(\eta) = y(x) = y(-\eta)$$

Then using the chain rule we can see that,

$$u'(\eta) = -y'(x) \qquad \text{and} \qquad u''(\eta) = y''(x)$$

With this transformation the differential equation becomes,

$$a(-\eta)^2 u'' + b(-\eta)(-u') + cu = 0$$

$$a\eta^2 u'' + b\eta u' + cu = 0$$

In other words, since $\eta>0$ we can use the work above to get solutions to this differential equation. We'll also go back to x's by using the variable transformation in reverse.

$$\eta = -x$$

Let's just take the real, distinct case first to see what happens.

$$u(\eta) = c_1 \eta^{r_1} + c_2 \eta^{r_2}$$

$$y(x) = c_1 (-x)^{r_1} + c_2 (-x)^{r_2}$$

Now, we could do this for the rest of the cases if we wanted to, but before doing that let's notice that if we recall the definition of absolute value,

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

we can combine both of our solutions to this case into one and write the solution as,

$$y(x) = c_1 |x|^{r_1} + c_2 |x|^{r_2}, \qquad x \ne 0$$

Note that we still need to avoid $x=0$ since we could still get division by zero. However this is now a solution for any interval that doesn't contain $x=0$.

We can do likewise for the other two cases and the following solutions for any interval not containing $x=0$,.

$$y(x) = c_1 |x|^r + c_2 |x|^r \ln|x|$$
$$y(x) = c_1 |x|^\lambda \cos(\mu \ln|x|) + c_2 |x|^\lambda \sin(\mu \ln|x|)$$

We can make one more generalization before working one more example. A more general form of an Euler Equation is,

$$a(x - x_0)^2 y'' + b(x - x_0) y' + cy = 0$$

and we can ask for solutions in any interval not containing $x = x_0$. The work for generating the solutions in this case is identical to all the above work and so isn't shown here.

The solutions in this general case for any interval not containing $x=a$ are,

$$y(x) = c_1 |x - a|^{r_1} + c_2 |x - a|^{r_2}$$
$$y(x) = |x - a|^r (c_1 + c_2 \ln|x - a|)$$
$$y(x) = |x - a|^\lambda (c_1 \cos(\mu \ln|x - a|) + c_2 \sin(\mu \ln|x - a|))$$

Where the roots are solutions to

$$ar(r-1) + b(r) + c = 0$$

Example 4 Find the solution to the following differential equation on any interval not containing x=*-6*.

$$3(x+6)^2 y'' + 25(x+6) y' - 16y = 0$$

Solution

So we get the roots from the identical quadratic in this case.

$$3r(r-1) + 25r - 16 = 0$$
$$3r^2 + 22r - 16 = 0$$
$$(3r-2)(r+8) = 0 \qquad \Rightarrow \qquad r_1 = \frac{2}{3},\ r_2 = -8$$

The general solution is then,

$$y(x) = c_1 |x - a|^{\frac{2}{3}} + c_2 |x - a|^{-8}$$

EX: Nullspace = Solution space to homogen diff linear eq'n

Defn: A set V together with two operations, called addition and scalar multiplication is a **vector space** if the following vector space axioms are satisfied for all vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V and all scalars, c, d in R.

Vector space axioms:

a.) $\mathbf{u} + \mathbf{v}$ is in V

b.) $c\mathbf{u}$ is in V

c.) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

d.) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

e.) There is a vector, denoted by $\mathbf{0}$, in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u}$ in V

f.) For each $\mathbf{u}$ in V, there is an element, denoted by $-\mathbf{u}$, in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

g.) $(cd)\mathbf{u} = c(d\mathbf{u})$

h.) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

i.) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

j.) $1\mathbf{u} = \mathbf{u}$

Examples:

1.) R^k with the usual operations of addition and scalar multiplication is a vector space.

2.) The set $M^{k,n}$, the set of all $k \times n$ matrices with the usual operations of addition and scalar multiplication is a vector

Linear Algebra Review: Eigenvalues and Eigenvectors

Defn: λ is an **eigenvalue** of the linear transformation $T : V \to V$ if there exists a nonzero vector $\mathbf{x}$ in V such that $T(\mathbf{x}) = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is said to be an **eigenvector** corresponding to the eigenvalue λ.

Example: Let $T(\mathbf{x}) = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}\mathbf{x}$. $Ax = \lambda x$

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix} = -1\begin{bmatrix} -1 \\ 5 \end{bmatrix}$

Thus -1 is an eigenvalue of A and $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus 5 is an eigenvalue of A and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix} \neq k\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ for any k.

Thus $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ is NOT an eigenvector of A.

MOTIVATION:

Note $\begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus $A \begin{bmatrix} 2 \\ 8 \end{bmatrix} = A(\begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}) = A \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3A \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$= -1 \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \cdot 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix}$$

Finding eigenvalues:

Suppose $A\mathbf{x} = \lambda\mathbf{x}$ (Note A is a SQUARE matrix).

Then $A\mathbf{x} = \lambda I\mathbf{x}$ where I is the identity matrix.

Thus $\lambda I\mathbf{x} - A\mathbf{x} = (\lambda I - A)\mathbf{x} = \mathbf{0}$

Thus if $A\mathbf{x} = \lambda\mathbf{x}$ for a nonzero $\mathbf{x}$, then $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has a nonzero solution.

Thus $det(\lambda I - A)\mathbf{x} = 0$.

Note that the eigenvectors corresponding to λ are the nonzero solutions of $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

Thus to find the eigenvalues of A and their corresponding eigenvectors:

Step 1: Find eigenvalues: Solve the equation

$$det(\lambda I - A) = 0 \text{ for } \lambda.$$

Step 2: For each eigenvalue λ_0, find its corresponding eigenvectors by solving the homogeneous system of equations

$$(\lambda_0 I - A)\mathbf{x} = 0 \text{ for } \mathbf{x}.$$

Defn: $det(\lambda I - A) = 0$ is the **characteristic equation** of A.

Thm 3: The eigenvalues of an upper triangular or lower triangular matrix (including diagonal matrices) are identical to its diagonal entries.

Defn: The **eigenspace** corresponding to an eigenvalue λ_0 of a matrix A is the set of all solutions of $(\lambda_0 I - A)\mathbf{x} = \mathbf{0}$.

Nullspace

Note: An eigenspace is a vector space

The vector $\mathbf{0}$ is always in the eigenspace.

The vector $\mathbf{0}$ is never an eigenvector. $A\vec{0} = \lambda\vec{0}$ for all λ

The number 0 can be an eigenvalue. need unique λ

Thm: A square matrix is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

Thus $\vec{0}$ not an e. vector

Ch 3/4

$$y^{(5)} + 12y^{(4)} + 104y^{(3)} + 408y'' + 1156y' = 0$$

Solution

The characteristic equation is,

$$r^5 + 12r^4 + 104r^3 + 408r^2 + 1156r = r\left(r^2 + 6r + 34\right)^2 = 0$$

So, we have one real root $r = 0$ and a pair of complex roots $r = -3 \pm 5i$ each with multiplicity 2. So, the solution for the real root is easy and for the complex roots we'll get a total of 4 solutions, 2 will be the *normal* solutions and two will be the normal solution each multiplied by t.

The general solution is,

$$y(t) = c_1 + c_2 e^{-3t}\cos(5t) + c_3 e^{-3t}\sin(5t) + c_4\, t\, e^{-3t}\cos(5t) + c_5\, t\, e^{-3t}\sin(5t)$$

Let's now work an example that contains all three of the basic cases just to say that we that we've got one work here.

linear homog

Example 4 Solve the following differential equation. Guess: $y = e^{rt}$

$$y^{(5)} - 15y^{(4)} + 84y^{(3)} - 220y'' + 275y' - 125y = 0$$

$r^5 e^{rt}$ $r^4 e^{rt}$ $r^3 e^{rt}$ $r^2 e^{rt}$ $r e^{rt}$ e^{rt}

Solution

The characteristic equation is

$$r^5 - 15r^4 + 84r^3 - 220r^2 + 275r - 125 = (r-1)(r-5)^2\left(r^2 - 4r + 5\right) = 0$$

$r = 1,\ r = 5,\ r = \dfrac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

In this case we've got one real distinct root, $r = 1$, and double root, $r = 5$, and a pair of complex roots, $r = 2 \pm i$ that only occur once.

The general solution is then,

$$y(t) = c_1 e^{t} + c_2 e^{5t} + c_3\, t\, e^{5t} + c_4 e^{2t}\cos(t) + c_5 e^{2t}\sin(t)$$

IVP:
$y(t_0) = y_0$
$y'(t_0) = y_1$
$y''(t_0) = y_2$
$y'''(t_0) = y_3$
$y^{iv}(t_0) = y_4$

We've got one final example to work here that on the surface at least seems almost too easy. The problem here will be finding the roots as well see.

Example 5 Solve the following differential equation.

$$y^{(4)} + 16y = 0$$

Solution

The characteristic equation is

$$r^4 + 16 = 0$$

So, a really simple characteristic equation. However, in order to find the roots we need to compute the fourth root of -16 and that is something that most people haven't done at this point in their mathematical career. We'll just give the formula here for finding them, but if you're interested in seeing a little more about this you might want to check out the Powers and Roots section of my Complex Numbers Primer.

The 4 (and yes there are 4!) 4^{th} roots of -16 can be found by evaluating the following,

$$\sqrt[4]{-16} = (-16)^{\frac{1}{4}} = \sqrt[4]{16}\, e^{\left(\frac{\pi}{4} + \frac{\pi k}{2}\right)i} = 2\left(\cos\left(\tfrac{\pi}{4} + \tfrac{\pi k}{2}\right) + i\sin\left(\tfrac{\pi}{4} + \tfrac{\pi k}{2}\right)\right) \qquad k = 0,1,2,3$$

Paul's online notes

or /Classes/DE/HOHomogeneousDE.asp

Note that each value of k will give a distinct 4^{th} root of -16. Also, note that for the 4^{th} root (and ONLY the 4^{th} root) of any negative number all we need to do is replace the 16 in the above formula with the absolute value of the number in question and this formula will work for those as well.

Here are the 4^{th} roots of -16.

$$k=0:\quad 2\left(\cos\left(\tfrac{\pi}{4}\right)+i\sin\left(\tfrac{\pi}{4}\right)\right)=2\left(\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i\right)=\sqrt{2}+i\sqrt{2}$$

$$k=1:\quad 2\left(\cos\left(\tfrac{3\pi}{4}\right)+i\sin\left(\tfrac{3\pi}{4}+\tfrac{\pi k}{2}\right)\right)=2\left(-\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i\right)=-\sqrt{2}+i\sqrt{2}$$

$$k=2:\quad 2\left(\cos\left(\tfrac{5\pi}{4}\right)+i\sin\left(\tfrac{5\pi}{4}\right)\right)=2\left(-\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}i\right)=-\sqrt{2}-i\sqrt{2}$$

$$k=3:\quad 2\left(\cos\left(\tfrac{7\pi}{4}\right)+i\sin\left(\tfrac{7\pi}{4}\right)\right)=2\left(\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}i\right)=\sqrt{2}-i\sqrt{2}$$

So, we have two sets of complex roots : $r=\sqrt{2}\pm i\sqrt{2}$ and $r=-\sqrt{2}\pm i\sqrt{2}$. The general solution is,

$$y(t)=c_1\mathbf{e}^{\sqrt{2}t}\cos\left(\sqrt{2}\,t\right)+c_2\mathbf{e}^{\sqrt{2}t}\sin\left(\sqrt{2}\,t\right)+c_3\mathbf{e}^{-\sqrt{2}t}\cos\left(\sqrt{2}\,t\right)+c_4\mathbf{e}^{-\sqrt{2}t}\sin\left(\sqrt{2}\,t\right)$$

So, we've worked a handful of examples here of higher order differential equations that should give you a feel for how these work in most cases.

There are of course a great many different kinds of combinations of the basic cases than what we did here and of course we didn't work any case involving 6^{th} order or higher, but once you've got an idea on how these work it's pretty easy to see that they all work pretty in pretty much the same manner. The biggest problem with the higher order differential equations is that the work in solving the characteristic polynomial and the system for the coefficients on the solution can be quite involved.

4.1: General Theory of nth Order Linear Equations

Thm: $L(y) = y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y$ is a linear function.

Proof: Let a, b be real numbers.

$L(af+bg)$

$= (af+bg)^{(n)} + p_1(t)(af+bg)^{(n-1)} + ... + p_{n-1}(t)(af+bg)' + p_n(t)(af+bg)$

$= af^{(n)} + bg^{(n)} + p_1(af^{(n-1)} + bg^{(n-1)}) + ... + p_{n-1}(af' + bg') + p_n(af+bg)$

$= af^{(n)} + p_1af^{(n-1)} + ... + p_{n-1}af' + p_naf + bg^{(n)} + p_1bg^{(n-1)} + ... + p_{n-1}bg' + p_nbg$

$= a[f^{(n)} + p_1f^{(n-1)} + ... + p_{n-1}f' + p_nf] + b[g^{(n)} + p_1g^{(n-1)} + ... + p_{n-1}g' + p_ng]$

$= aL(f) + bL(g)$

Theorem: If ϕ_i, $i = 1, ..., n$ are solutions to a homogeneous linear differential equation (i.e., $y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0$ (*)), then $c_1\phi_1 + c_2\phi_2 + ... + c_n\phi_n$ is also a solution to this linear differential equation.

Pf: Since ϕ_i, $i = 1, ..., n$ are solutions to (*), $L(\phi_i) = 0$ for $i = 1, ..., n$.
Thus $L(c_1\phi_1 + c_2\phi_2 + ... + c_n\phi_n) = c_1L(\phi_1) + c_2L(\phi_2) + ... + c_nL(\phi_n) = 0$.
Thus $c_1\phi_1 + c_2\phi_2 + ... + c_n\phi_n$ is also a solution to (*).

Solve: $y'' + y = 0$, $y(0) = -1$, $y'(0) = -3$

$r^2 + 1 = 0$ implies $r^2 = -1$. Thus $r = \pm i$.

Since $r = 0 \pm 1i$, $y = k_1cos(t) + k_2sin(t)$. Then $y' = -k_1sin(t) + k_2cos(t)$

$y(0) = -1$: $\quad -1 = k_1cos(0) + k_2sin(0)$ implies $-1 = k_1$

$y'(0) = -3$: $\quad -3 = -k_1sin(0) + k_2cos(0)$ implies $-3 = k_2$

Thus IVP solution: $y = -cos(t) - 3sin(t)$

When does the following IVP have a unique solution:

IVP: $y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0$,

$$y(t_0) = y_0,\ y'(t_0) = y_1, ..., y^{(n-1)}(t_0) = y_{n-1}.$$

Suppose $y = c_1\phi_1(t) + c_2\phi_2(t) + ... + c_n\phi_n(t)$ is a solution to this IVT Then

$$y(t_0) = y_0: \quad y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0) + ... + c_n\phi_n(t_0)$$

$$y'(t_0) = y_1: \quad y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0) + ... + c_n\phi_n'(t_0)$$

$$\vdots$$

$$y^{(n-1)}(t_0) = y_{n-1}: \quad y_{n-1} = c_1\phi_1^{(n-1)}(t_0) + c_2\phi_2^{(n-1)}(t_0) + ... + c_n\phi_n^{(n-1)}(t_0)$$

To find IVP solution, need to solve above system of equations for the unknowns c_i, $i = 1, ..., n$.

Note the IVP has a unique solution if and only if the above system of equations has a unique solution for the c_i.

Note that in these equations the c_i are the unknowns and the $y_i, \phi_i(t_0), ...\phi_i^{(n-1)}(t_0)$, are the constants.
We can translate this linear system of equations into matrix form:

$$\begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & ... & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & ... & \phi_n'(t_0) \\ & \vdots & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & ... & \phi_n^{(n-1)}(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

Note this equation has a unique solution if and only if

$$det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & ... & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & ... & \phi_n'(t_0) \\ & \vdots & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & ... & \phi_n^{(n-1)}(t_0) \end{bmatrix} \neq 0$$

Definition: The Wronskian of the functions, $\phi_1, \phi_2, ..., \phi_n$ is

$$W(\phi_1, \phi_2, ..., \phi_n) = det \begin{bmatrix} \phi_1(t) & \phi_2(t) & ... & \phi_n(t) \\ \phi_1'(t) & \phi_2'(t) & ... & \phi_n'(t) \\ & \vdots & & \\ \phi_1^{(n-1)}(t) & \phi_2^{(n-1)}(t) & ... & \phi_n^{(n-1)}(t) \end{bmatrix}$$

Theorem: Suppose that ϕ_i, $i = 1, ..., n$ are solutions to

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0.$$

If $W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0$, then there is a unique choice of constants c_i such that $c_1\phi_1 + c_2\phi_2 + ... + c_n\phi_n$ satisfies this homogeneous linear differential equation and initial conditions, $y(t_0) = y_0$, $y'(t_0) = y_1, ..., y^{(n-1)}(t_0) = y_{n-1}$.

Recall $\phi_1, ..., \phi_n$ are linearly independent iff $c_1 = ... = c_n = 0$ is the only solution to $c_1\phi_1 + ... + c_n\phi_n = \mathbf{0}$.

If ϕ_i are functions of t, then $\mathbf{0}$ is the constant function, $\mathbf{0}(t) = 0$ for all t. Thus $c_1\phi_1(t) + ... + c_n\phi_n(t) = 0$ for all t.

Hence $c_1\phi_1^{(k)}(t) + ... + c_n\phi_n^{(k)}(t) = 0$ for all t, k if derivatives exist.

Thus $\phi_1, ..., \phi_n$ are linearly independent iff for any given f, $c_1\phi_1(t) + ... + c_n\phi_n(t) = 0$ has a unique solution (that works for all t).

iff the following system of equations has a unique solution

$$c_1\phi_1(t) + c_2\phi_2(t) + ... + c_n\phi_n(t) = 0$$
$$c_1\phi_1'(t) + c_2\phi_2'(t) + ... + c_n\phi_n'(t) = 0$$
$$\vdots$$
$$c_1\phi_1^{(n-1)}(t) + c_2\phi_2^{(n-1)}(t) + ... + c_n\phi_n^{(n-1)}(t) = 0$$

iff the following system of equations has a unique solution

$$\begin{bmatrix} \phi_1(t) & \phi_2(t) & ... & \phi_n(t) \\ \phi_1'(t) & \phi_2'(t) & ... & \phi_n'(t) \\ & \vdots & & \\ \phi_1^{(n-1)}(t) & \phi_2^{(n-1)}(t) & ... & \phi_n^{(n-1)}(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Note this equation has a unique solution if and only if for some t_0

$$det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & ... & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & ... & \phi_n'(t_0) \\ & \vdots & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & ... & \phi_n^{(n-1)}(t_0) \end{bmatrix} \neq 0$$

iff $\quad W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0$,

Example: Determine if $\{1+2t, 5+4t^2, 6-8t+8t^2\}$ are linearly independent:

Method 1: Solve $c_1(1+2t) + c_2(5+4t^2) + c_3(6-8t+8t^2) = 0$

Or equivalently, solve $c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 6 \\ -8 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Or equivalently, solve $\begin{bmatrix} 1 & 5 & 6 \\ 2 & 0 & -8 \\ 0 & 4 & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Method 2: Check the Wronskian

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nth order LINEAR differential equation:

Theorem 4.1.1: If $p_i : (a,b) \to R$, $i = 1, ..., n$ and $g : (a,b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a,b) \to R$ that satisfies the initial value problem

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = g(t),$$
$$y(t_0) = y_0, \quad y'(t_0) = y_1, \quad y^{(n-1)}(t_0) = y_{n-1}$$

Proof: We proved the case $n = 1$ using an integrating factor. When $n > 1$, see more advanced textbook.

Claim: If p_i are continuous on (a,b), if $\phi_1, ..., \phi_n$ are linearly independent solutions to

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0,$$

then $\{\phi_1, ..., \phi_n\}$ is a basis for the solution set to this differential equation.

Theorem 4.1.2: If p_i are continuous on (a,b), suppose that ϕ_i, $i = 1, ..., n$ are solutions to $y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0$.
If $W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0$, for some $t_0 \in (a,b)$, then any solution to this homogeneous linear differential equation can be written as

$$y = c_1\phi_1 + c_2\phi_2 + ... + c_n\phi_n \text{ for some constants } c_i.$$

Defn: The $\phi_1, ..., \phi_n$ are called a fundamental set of solutions to

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0.$$

Theorem: Given any nth order homogeneous linear differential equation, there exist a set of n functions which form a fundamental set of solutions.