5.1 Review of Power Series.

Definition: $\Sigma_{n=0}^{\infty} a_n(x-x_0)^n = lim_{n\to\infty} \Sigma_{n=0}^{k} a_n(x-x_0)^n$

Taylor's Theorem

Suppose f has $n+1$ continuous derivatives on an open interval containing a. Then for each x in the interval,

$$f(x) = \left[\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k\right] + R_{n+1}(x)$$

where the error term $R_{n+1}(x)$ satisfies $R_{n+1}(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x.

The *infinite* Taylor series converges to f,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \text{ if and only if } \lim_{n\to\infty} R_n(x) = 0.$$

Defn: The function f is said to be **analytic** at a is its Taylor series expansion about $x=a$ has a positive radius of convergence.

1.) $\Sigma_{n=0}^{\infty} a_n(x-x_0)^n$ converges at the point x if and only if $lim_{n\to\infty} \Sigma_{n=0}^{k} a_n(x-x_0)^n$ exists at the point x.

2.) $\Sigma_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely at the point x if and only if $\Sigma_{n=0}^{\infty}|a_n||x-x_0|^n$ converges at the point x

If a series converges absolutely, then it also converges.

3.) Ratio test for absolute convergence:

Let $L = lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|$

$lim_{n\to\infty}|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}| = |x-x_0| lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = |x-x_0|L < 1$

The power series converges at the value x if $|x-x_0| < \frac{1}{L}$

The power series diverges at the value x if $|x-x_0| > \frac{1}{L}$

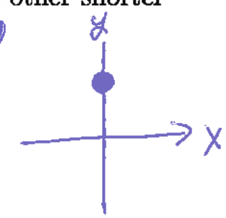
The ratio test give no info at the value x if $|x-x_0| = \frac{1}{L}$ ← check endpts if finite radius of convergence

Note $\frac{1}{L}$ is the **radius of convergence**.

Solve $y'' - 4y' + 4y = 0$

Using quick 3.4 method. Guess $y = e^{rt}$ and plug into equation to find $r^2 - 4r + 4 = 0$. Thus $(r-2)^2 = 0$. Hence $r = 2$. Therefore general solution is $y = c_1 e^{2x} + c_2 x e^{2x}$.

Use LONG 5.2 method (normally use this method only when other shorter methods don't exist) to find solution for values near $x_0 = 0$.



Suppose the solution $y = f(x)$ is analytic at $x_0 = 0$.

That is $f(x) = \Sigma_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n$ for x near $x_0 = 0$.

Thus there are constants $a_n = \frac{f^{(n)}(0)}{n!}$ such that,

$$f(x) = \Sigma_{n=0}^{\infty} a_n (x-0)^n = \Sigma_{n=0}^{\infty} a_n x^n.$$

Find a recursive formula for the constants of the series solution to $y'' - 4y' + 4y = 0$ near $x_0 = 0$

We will determine these constants a_n by plugging f into the ODE.

$f(x) = \Sigma_{n=0}^{\infty} a_n x^n$, $f'(x) = \Sigma_{n=1}^{\infty} a_n n x^{n-1}$, $f''(x) = \Sigma_{n=2}^{\infty} a_n n(n-1)x^{n-2}$.

$\Sigma_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 4\Sigma_{n=1}^{\infty} a_n n x^{n-1} + 4\Sigma_{n=0}^{\infty} a_n x^n = 0$.

$\Sigma_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - 4\Sigma_{n=0}^{\infty} a_{n+1}(n+1)x^n + 4\Sigma_{n=0}^{\infty} a_n x^n = 0$.

$\Sigma_{n=0}^{\infty} [a_{n+2}(n+2)(n+1) - 4a_{n+1}(n+1) + 4a_n] x^n = 0$.

$a_{n+2}(n+2)(n+1) - 4a_{n+1}(n+1) + 4a_n = 0$.

$a_{n+2} = \frac{4a_{n+1}(n+1) - 4a_n}{(n+2)(n+1)}$.

Solve for term w/ highest subscript

Hence the recursive formula (if know previous terms, can determine later terms) is

$$a_{n+2} = 4\left(\frac{(n+1)a_{n+1} - a_n}{(n+2)(n+1)}\right)$$

Given the recursive formula, $a_{n+2} = 4\left(\frac{(n+1)a_{n+1} - a_n}{(n+2)(n+1)}\right)$, determine a_n.

Determine formula for a_k by noticing patterns. Note: It is easier to notice patterns if you do NOT simplify too much.

Find the first 6 terms of the series solution

$n = 0: \quad a_2 = 4\left(\frac{a_1 - a_0}{(2)(1)}\right)$

$n = 1: \quad a_3 = 4\left(\frac{2a_2 - a_1}{(3)(2)}\right) = 4\left(\frac{(2)(4)\left(\frac{a_1 - a_0}{(2)(1)}\right) - a_1}{(3)(2)}\right) = 4\left(\frac{4(a_1 - a_0) - a_1}{(3)(2)}\right)$

$= 4\left(\frac{3a_1 - 4a_0}{3!}\right)$

$n = 2: \quad a_4 = 4\left(\frac{3a_3 - a_2}{(4)(3)}\right) = 4\left(\frac{3(4)\left(\frac{3a_1 - 4a_0}{3!}\right) - 4\left(\frac{a_1 - a_0}{2!}\right)}{(4)(3)}\right) = 4\left(\frac{3\left(\frac{3a_1 - 4a_0}{3!}\right) - \left(\frac{a_1 - a_0}{2!}\right)}{3}\right)$

$= 4\left(\frac{\left(\frac{3a_1 - 4a_0}{2!}\right) - \left(\frac{a_1 - a_0}{2!}\right)}{3}\right) = 4\left(\frac{(3a_1 - 4a_0) - (a_1 - a_0)}{3!}\right) = 4\left(\frac{2a_1 - 3a_0}{(3!)}\right)$

$n = 3: \quad a_5 = 4\left(\frac{(4)a_4 - a_3}{(5)(4)}\right) = 4\left(\frac{(4)4\left(\frac{2a_1 - 3a_0}{3!}\right) - 4\left(\frac{3a_1 - 4a_0}{3!}\right)}{(5)(4)}\right)$

$= 4\left(\frac{4\left(\frac{2a_1 - 3a_0}{3!}\right) - \left(\frac{3a_1 - 4a_0}{3!}\right)}{5}\right) = 4\left(\frac{4(2a_1 - 3a_0) - (3a_1 - 4a_0)}{5(3!)}\right) = 4\left(\frac{5a_1 - 8a_0}{5(3!)}\right)$

$f(x) \sim a_0 + a_1 x + 4\left(\frac{a_1 - a_0}{2!}\right)x^2 + 4\left(\frac{3a_1 - 4a_0}{3!}\right)x^3 + 4\left(\frac{2a_1 - 3a_0}{(3!)}\right)x^4 + 4\left(\frac{5a_1 - 8a_0}{5(3!)}\right)x^5$

Recall $f(x) = a_0\phi_0(x) + a_1\phi_1(x)$ for linearly independent solutions ϕ_0 and ϕ_1 to equation $y'' - 4y' + 4y = 0$.

Find the first 5 terms in each of the 2 solns $y = \phi_0(x)$ and $y = \phi_1(x)$

$\phi_0 \sim 1 + 4\left(\frac{-1}{2!}\right)x^2 + 4\left(\frac{-4}{3!}\right)x^3 + 4\left(\frac{-3}{(3!)}\right)x^4 + 4\left(\frac{-8}{5(3!)}\right)x^5$

$\phi_1 \sim x + 4\left(\frac{1}{2!}\right)x^2 + 4\left(\frac{3}{3!}\right)x^3 + 4\left(\frac{2}{(3!)}\right)x^4 + 4\left(\frac{5}{5(3!)}\right)x^5$

$n = 0: \quad a_2 = 4\left(\frac{a_1 - a_0}{(2)(1)}\right) = 2\left(\frac{2a_1 - 2a_0}{2!}\right)$

$n = 1: \quad a_3 = 4\left(\frac{3a_1 - 4a_0}{3!}\right) = 2^2\left(\frac{3a_1 - 4a_0}{3!}\right)$

$n = 2: \quad a_4 = 4\left(\frac{2a_1 - 3a_0}{3!}\right) = 16\left(\frac{2a_1 - 3a_0}{4!}\right) = 8\left(\frac{4a_1 - 6a_0}{4!}\right) = 2^3\left(\frac{4a_1 - 6a_0}{4!}\right)$

$n = 3: \quad a_5 = 4\left(\frac{5a_1 - 8a_0}{5(3!)}\right) = 16\left(\frac{5a_1 - 8a_0}{5!}\right) = 2^4\left(\frac{5a_1 - 8a_0}{5!}\right)$

Hence it appears $a_k = \frac{2^{k-1}(ka_1 - 2(k-1)a_0)}{k!}$

2 looking for $n!$

Found a_k in terms of 2 unknowns a_0 & a_1

hypothesis

what we need to prove

Prove that if $a_{n+2} = 4\left(\frac{(n+1)a_{n+1}-a_n}{(n+2)(n+1)}\right)$, then $a_k = \frac{2^{k-1}(ka_1-2(k-1)a_0)}{k!}$

Need to prove $a_k = \frac{2^{k-1}(ka_1-2(k-1)a_0)}{k!}$ for $k \geq 0$

Given: $a_{n+2} = 4\left(\frac{(n+1)a_{n+1}-a_n}{(n+2)(n+1)}\right)$ for $n \geq 2$,

$k=0$

Proof by induction on k.

base case

Suppose $k = 0$. Then $\frac{2^{0-1}(0(a_1)-2(-1)a_0)}{0!} = \frac{1}{2}(2a_0) = a_0$

$k=1$

Suppose $k = 1$. Then $\frac{2^{1-1}(1(a_1)-2(1-1)a_0)}{1!} = a_1$

Suppose $a_k = \frac{2^{k-1}(ka_1-2(k-1)a_0)}{k!}$ for $k = n, n+1$

induction hypothesis

helps to be specific

Thus $a_n = \frac{2^{n-1}(na_1-2(n-1)a_0)}{n!}$ and $a_{n+1} = \frac{2^n((n+1)a_1-2na_0)}{(n+1)!}$

Claim: $a_{n+2} = \frac{2^{n+1}((n+2)a_1-2(n+1)a_0)}{(n+2)!}$

Need to prove

using induction hyp

$$a_{n+2} = 4\left(\frac{(n+1)a_{n+1}-a_n}{(n+2)(n+1)}\right) = 4\left(\frac{(n+1)\left[\frac{2^n((n+1)a_1-2na_0)}{(n+1)!}\right]-\left[\frac{2^{n-1}(na_1-2(n-1)a_0)}{n!}\right]}{(n+2)(n+1)}\right)$$

use hypothesis

$$= 4\left(\frac{\left[\frac{2^n((n+1)a_1-2na_0)}{n!}\right]-\left[\frac{2^{n-1}(na_1-2(n-1)a_0)}{n!}\right]}{(n+2)(n+1)}\right)$$

$$= 4(2)^{n-1}\left(\frac{[2((n+1)a_1-2na_0)]-[na_1-2(n-1)a_0]}{n!(n+2)(n+1)}\right)$$

$$= 2^{n+1}\left(\frac{2(n+1)a_1-4na_0-na_1+2(n-1)a_0}{n!(n+2)(n+1)}\right) = 2^{n+1}\left(\frac{(n+2)a_1-2(n+1)a_0}{(n+2)!}\right) \quad \square$$

Thus $f(x) = \Sigma_{n=0}^{\infty}\frac{2^{n-1}(na_1-2(n-1)a_0)}{n!}x^n$

$$= a_1\Sigma_{n=0}^{\infty}\frac{2^{n-1}(n)}{n!}x^n - 2a_0\Sigma_{n=0}^{\infty}\frac{2^{n-1}(n-1)}{n!}x^n$$

$$= a_0(-2)\Sigma_{n=0}^{\infty}\frac{2^{n-1}(n-1)}{n!}x^n + a_1\Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^n$$

if these two series converge.

$$\sum_{n=0}^{\infty}\frac{2^{n-1}(n)a_1x^n}{n!} + \sum_{n=0}^{\infty}\frac{2^{n-1}(-2(n-1))a_0x^n}{n!}$$

For what values of x does $\Sigma_{n=0}^{\infty}\frac{(n-1)2^{n-1}}{n!}x^n$ converge

Ratio test: Suppose we have the series Σb_n . Let $L = lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right|$

Then, if $L < 1$, the series is absolutely convergent (and hence convergent).

If $L > 1$, the series is divergent.

If $L = 1$, the series may be divergent, conditionally convergent, or absolutely convergent.

$$lim_{n\to\infty}\left|\frac{\frac{n2^n}{(n+1)!}x^{n+1}}{\frac{(n-1)2^{n-1}}{n!}x^n}\right| = lim_{n\to\infty}\left|\frac{2nx}{(n+1)(n-1)}\right|$$

$$= 2x\ lim_{n\to\infty}\left|\frac{n}{(n+1)(n-1)}\right| = 0 < 1$$

Hence the series converges for all x

For what values of x does $\Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^n$ converge

$$lim_{n\to\infty}\left|\frac{\frac{2^n}{n!}x^{n+1}}{\frac{2^{n-1}}{(n-1)!}x^n}\right| = lim_{n\to\infty}\left|\frac{2x}{n}\right| = 2x\ lim_{n\to\infty}\left|\frac{1}{n}\right| = 0 < 1$$

Hence the series converges for all x

Thus the solution is

$$f(x) = a_0(-2)\Sigma_{n=0}^{\infty}\frac{2^{n-1}(n-1)}{n!}x^n + a_1\Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^n$$

and the domain is all real numbers.

I.e., the general solution is $f(x) = a_0\phi_0(x) + a_1\phi_1(x)$

where $\phi_0(x) = (-2)\Sigma_{n=0}^{\infty}\frac{2^{n-1}(n-1)}{n!}x^n$ and $\phi_1(x) = \Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^n$

Note we could have replaced the constant a_0 with $-2a_0$, but the a_i's have meaning: $a_n = \frac{f^{(n)}(0)}{n!}$. Thus our initial values are $a_0 = f(0)$ and $a_1 = f'(0)$

If works $a_0 = f(0)$, $a_1 = f'(0)$ since $a_n = \frac{f^{(n)}(0)}{n!}$

In general, to determine if there is a unique solution to the IVP, $y'' - 4y' + 4y = 0$, $y(x_0) = y_0$, $y'(x_0) = y_1$, we solve for unknowns a_0 and a_1.

$y(x_0) = a_0\phi_0(x_0) + a_1\phi_1(x_0)$
$y'(x_0) = a_0\phi_0'(x_0) + a_1\phi_1'(x_0)$

Note that the above system of two equations has a unique solution for the two unknowns a_0 and a_1 if and only if $det\begin{pmatrix}\phi_0(x_0) & \phi_1(x_0)\\ \phi_0'(x_0) & \phi_1'(x_0)\end{pmatrix} \neq 0$

In other words the IVP has a unique solution iff the Wronskian of ϕ_0 and ϕ_1 evaluated at x_0 is not zero. Recall that by theorem , this also implies that ϕ_0 and ϕ_1 are linearly independent and hence the general solution is $y = a_0\phi_0(x) + a_1\phi_1(x)$ by theorem.

Show that $\phi_0(x) = (-2)\Sigma_{n=0}^{\infty}\frac{2^{n-1}((n-1))}{n!}x^n$ and $\phi_1(x) = \Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^n$ are linearly independent by calculating the Wronskian of these two functions evaluated at $x_0 = 0$.

$$W(\phi_1,\phi_2)(x) = \begin{pmatrix}\phi_1(x) & \phi_2(x)\\ \phi_1'(x) & \phi_2'(x)\end{pmatrix} = \begin{pmatrix}(-2)\Sigma_{n=0}^{\infty}\frac{2^{n-1}(n-1)}{n!}x^n & \Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^n\\ (-2)\Sigma_{n=1}^{\infty}\frac{2^{n-1}(n-1)}{(n-1)!x^{n-1}} & \Sigma_{n=1}^{\infty}\frac{n2^{n-1}}{(n-1)!}x^{n-1}\end{pmatrix}$$

$$W(\phi_1,\phi_2)(0) = \begin{pmatrix}(-2)2^{0-1}(-1) & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 1 \neq 0$$

Hence $\phi_0(x) = (-2)\Sigma_{n=0}^{\infty}\frac{2^{n-1}((n-1))}{n!}x^n$ and $\phi_1(x) = \Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^n$ are linearly independent

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When possible identify the functions giving the series solutions. Recall that by Taylor's theorem and the ratio test, $e^{2x} = \Sigma_{n=0}^{\infty}\frac{f^{(n)}(x)}{n!}x^n = \Sigma_{n=0}^{\infty}\frac{2^n}{n!}x^n$ for all x.

$f(x) = a_1\Sigma_{n=0}^{\infty}\frac{n2^{n-1}}{n!}x^n - 2a_0\Sigma_{n=0}^{\infty}\frac{2^{n-1}(n-1)}{n!}x^n$

$= a_1\Sigma_{n=0}^{\infty}\frac{n2^{n-1}}{n!}x^n - 2a_0\Sigma_{n=0}^{\infty}\frac{n2^{n-1}}{n!}x^n + 2a_0\Sigma_{n=0}^{\infty}\frac{2^{n-1}}{n!}x^n$

$= (a_1 - 2a_0)\Sigma_{n=0}^{\infty}\frac{n2^{n-1}}{n!}x^n + a_0\Sigma_{n=0}^{\infty}\frac{2^n}{n!}x^n$

$(a-2a_0)(\quad) + a_0(e^{2x})$

$= (a_1 - 2a_0)x\Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^{n-1} + a_0\Sigma_{n=0}^{\infty}\frac{2^n}{n!}x^n$

$= (a_1 - 2a_0)x\Sigma_{n=0}^{\infty}\frac{2^n}{n!}x^n + a_0\Sigma_{n=0}^{\infty}\frac{2^n}{n!}x^n$

$= (a_1 - 2a_0)(xe^{2x}) + a_0(e^{2x})$

Note we have recovered the solution we found using the 3.4 method.

Note a power series solutions exists in a neighborhood of x_0 when the solution is analytic at x_0. I.e, the solution is of the form $y = \Sigma_{n=0}^{\infty}a_n(x-x_0)^n$ where this series has a nonzero radius of convergence about x_0.

When do we know an analytic solution exists? I.e, when is this method guaranteed to work?

Special case: $P(x)y'' + Q(x)y' + R(x)y = 0$

Then $y''(x) = -\frac{Q}{P}y' - \frac{R}{P}y$

Definition: The point x_0 is an *ordinary point* of the ODE,

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

if $\frac{Q}{P}$ and $\frac{R}{P}$ are analytic at x_0.

Theorem 5.3.1: If x_0 is an ordinary point of the ODE $P(x)y'' + Q(x)y' + R(x)y = 0$, then the general solution to this ODE is

$$y = \Sigma_{n=1}^{\infty}a_n(x-x_0)^n = a_0\phi_0(x) + a_1\phi_1(x)$$

where ϕ_i are power series solutions that are analytic at x_0. The solutions ϕ_0, ϕ_1 form a fundamental set of solutions. The radius of convergence for each of these series solutions is at least as large as the minimum radii of convergence of the series for $\frac{Q}{P}$ and $\frac{R}{P}$.

Theorem: If P and Q are polynomial functions, then $y = Q(x)/P(x)$ is analytic at x_0 if and only if $P(x_0) \neq 0$. Moreover if Q/P is reduced, the radius of convergence of $Q(x)/P(x) = min\{||x_0 - x|| \mid x \in \mathbf{C}, P(x) = 0\}$ where $||x_0 - x||$ = distance from x_0 to x in the complex plane.