5.2

Solve $y'' - 4y' + 4y = 0$

Using quick 3.4 method. Guess $y = e^{rt}$ and plug into equation to find $r^2 - 4r + 4 = 0$. Thus $(r-2)^2 = 0$. Hence $r = 2$. Therefore general solution is $y = c_1e^{2x} + c_2xe^{2x}$.

Use LONG 5.2 method (normally use this method only when other shorter methods don't exist) to find solution for values near $x_0 = 0$.

in real life translate values initial to $x_0 = 0$

Suppose the solution $y = f(x)$ is analytic at $x_0 = 0$.

That is $\boxed{f(x) = \Sigma_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n}$ for x near $x_0 = 0$.

Thus there are constants $a_n = \frac{f^{(n)}(0)}{n!}$ such that,

$$\boxed{f(x) = \Sigma_{n=0}^{\infty} a_n(x-0)^n = \Sigma_{n=0}^{\infty} a_nx^n.}$$

assuming $y = \sum_0^\infty a_n x^n$

Find a recursive formula for the constants of the series solution to $y'' - 4y' + 4y = 0$ near $x_0 = 0$

We will determine these constants a_n by plugging f into the ODE.

$f(x) = \Sigma_{n=0}^{\infty} a_nx^n$, $f'(x) = \Sigma_{n=1}^{\infty} a_nnx^{n-1}$, $f''(x) = \Sigma_{n=2}^{\infty} a_nn(n-1)x^{n-2}$.

$\Sigma_{n=2}^{\infty} a_nn(n-1)x^{n-2} - 4\Sigma_{n=1}^{\infty} a_nnx^{n-1} + 4\Sigma_{n=0}^{\infty} a_nx^n = 0$.

$y'' \quad - 4y' \quad + 4y \quad = 0$

$\Sigma_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - 4\Sigma_{n=0}^{\infty} a_{n+1}(n+1)x^n + 4\Sigma_{n=0}^{\infty} a_nx^n = 0$.

$\Sigma_{n=0}^{\infty} [a_{n+2}(n+2)(n+1) - 4a_{n+1}(n+1) + 4a_n]x^n = 0$.

$a_{n+2}(n+2)(n+1) - 4a_{n+1}(n+1) + 4a_n = 0$.

$a_{n+2} = \frac{4a_{n+1}(n+1) - 4a_n}{(n+2)(n+1)}$.

Hence the recursive formula (if know previous terms, can determine later terms) is

$$a_{n+2} = 4\left(\frac{(n+1)a_{n+1} - a_n}{(n+2)(n+1)}\right)$$

$$a_k = 4\left(\frac{(k-1)(a_{k-1}) - (a_{k-2})}{k(k-1)}\right) =$$

Given the recursive formula, $a_{n+2} = 4\left(\frac{(n+1)a_{n+1} - a_n}{(n+2)(n+1)}\right)$, determine a_n.

Determine formula for a_k by noticing patterns. Note: It is easier to notice patterns if you do NOT simplify too much.

Find the first 6 terms of the series solution

$n = 0: \quad a_2 = 4\left(\frac{a_1 - a_0}{(2)(1)}\right)$

$n = 1: \quad a_3 = 4\left(\frac{2a_2 - a_1}{(3)(2)}\right) = 4\left(\frac{(2)(4)\left(\frac{a_1-a_0}{(2)(1)}\right) - a_1}{(3)(2)}\right) = 4\left(\frac{4(a_1-a_0)-a_1}{(3)(2)}\right)$

$= 4\left(\frac{3a_1 - 4a_0}{3!}\right)$

$n = 2: \quad a_4 = 4\left(\frac{3a_3 - a_2}{(4)(3)}\right) = 4\left(\frac{3(4)\left(\frac{3a_1-4a_0}{3!}\right) - 4\left(\frac{a_1-a_0}{2!}\right)}{(4)(3)}\right) = 4\left(\frac{3\left(\frac{3a_1-4a_0}{3!}\right) - \left(\frac{a_1-a_0}{2!}\right)}{3}\right)$

$= 4\left(\frac{\left(\frac{3a_1-4a_0}{2!}\right) - \left(\frac{a_1-a_0}{2!}\right)}{3}\right) = 4\left(\frac{(3a_1-4a_0)-(a_1-a_0)}{3!}\right) = 4\left(\frac{2a_1 - 3a_0}{(3!)}\right)$

$n = 3: \quad a_5 = 4\left(\frac{(4)a_4 - a_3}{(5)(4)}\right) = 4\left(\frac{(4)4\left(\frac{2a_1-3a_0}{3!}\right) - 4\left(\frac{3a_1-4a_0}{3!}\right)}{(5)(4)}\right)$

$= 4\left(\frac{4\left(\frac{2a_1-3a_0}{3!}\right) - \left(\frac{3a_1-4a_0}{3!}\right)}{5}\right) = 4\left(\frac{4(2a_1-3a_0)-(3a_1-4a_0)}{5(3!)}\right) = 4\left(\frac{5a_1 - 8a_0}{5(3!)}\right)$

$f(x) \sim a_0 + a_1x + 4\left(\frac{a_1-a_0}{2!}\right)x^2 + 4\left(\frac{3a_1-4a_0}{3!}\right)x^3 + 4\left(\frac{2a_1-3a_0}{(3!)}\right)x^4 + 4\left(\frac{5a_1-8a_0}{5(3!)}\right)x^5$

approx soln about x_0

Recall $f(x) = a_0\phi_0(x) + a_1\phi_1(x)$ for linearly independent solutions ϕ_0 and ϕ_1 to equation $y'' - 4y' + 4y = 0$.

Find the first 5 terms in each of the 2 solns $y = \phi_0(x)$ and $y = \phi_1(x)$

$\phi_0 \sim 1 + 4\left(\frac{-1}{2!}\right)x^2 + 4\left(\frac{-4}{3!}\right)x^3 + 4\left(\frac{-3}{(3!)}\right)x^4 + 4\left(\frac{-8}{5(3!)}\right)x^5$

$\phi_1 \sim x + 4\left(\frac{1}{2!}\right)x^2 + 4\left(\frac{3}{3!}\right)x^3 + 4\left(\frac{2}{(3!)}\right)x^4 + 4\left(\frac{5}{5(3!)}\right)x^5$

degree 5 polynomial approx of 2 homog. solns

$n = 0: \quad a_2 = 4\left(\frac{a_1 - a_0}{(2)(1)}\right) = 2\left(\frac{2a_1 - 2a_0}{2!}\right)$

$n = 1: \quad a_3 = = 4\left(\frac{3a_1 - 4a_0}{3!}\right) = 2^2\left(\frac{3a_1 - 4a_0}{3!}\right)$

$n = 2: \quad a_4 = 4\left(\frac{2a_1 - 3a_0}{3!}\right) = 16\left(\frac{2a_1 - 3a_0}{4!}\right) = 8\left(\frac{4a_1 - 6a_0}{4!}\right) = 2^3\left(\frac{4a_1 - 6a_0}{4!}\right)$

$n = 3: \quad a_5 = 4\left(\frac{5a_1 - 8a_0}{5(3!)}\right) = 16\left(\frac{5a_1 - 8a_0}{5!}\right) = 2^4\left(\frac{5a_1 - 8a_0}{5!}\right)$

Hence it appears $a_k = \frac{2^{k-1}(ka_1 - 2(k-1)a_0)}{k!}$

The best degree 5 polynomial approx to soln

S_5 identical

✓ Known recursive

believed

Prove that if $a_{n+2} = 4\left(\frac{(n+1)a_{n+1}-a_n}{(n+2)(n+1)}\right)$, then $a_k = \frac{2^{k-1}(ka_1-2(k-1)a_0)}{k!}$

Need to prove $a_k = \frac{2^{k-1}(ka_1-2(k-1)a_0)}{k!}$ for $k \geq 0$

Given: $a_{n+2} = 4\left(\frac{(n+1)a_{n+1}-a_n}{(n+2)(n+1)}\right)$ for $n \geq 2$, Know recursive formula

Proof by induction on k.

Suppose $k=0$. Then $\frac{2^{0-1}(0(a_1)-2(-1)a_0)}{0!} = \frac{1}{2}(2a_0) = a_0$

Suppose $k=1$. Then $\frac{2^{1-1}(1(a_1)-2(1-1)a_0)}{1!} = a_1$ Base case

Suppose $a_k = \frac{2^{k-1}(ka_1-2(k-1)a_0)}{k!}$ for $k = n, n+1$ Induction hypothesis

Thus $a_n = \frac{2^{n-1}(na_1-2(n-1)a_0)}{n!}$ and $a_{n+1} = \frac{2^n((n+1)a_1-2na_0)}{(n+1)!}$ simplified ⇐ ind hyp

Claim: $a_{n+2} = \frac{2^{n+1}((n+2)a_1-2(n+1)a_0)}{(n+2)!}$

$$a_{n+2} = 4\left(\frac{(n+1)a_{n+1}-a_n}{(n+2)(n+1)}\right) = 4\left(\frac{(n+1)\left[\frac{2^n((n+1)a_1-2na_0)}{(n+1)!}\right]-\left[\frac{2^{n-1}(na_1-2(n-1)a_0)}{n!}\right]}{(n+2)(n+1)}\right)$$

$$= 4\left(\frac{\left[\frac{2^n((n+1)a_1-2na_0)}{n!}\right]-\left[\frac{2^{n-1}(na_1-2(n-1)a_0)}{n!}\right]}{(n+2)(n+1)}\right)$$

$$= 4(2)^{n-1}\left(\frac{[2((n+1)a_1-2na_0)]-[na_1-2(n-1)a_0]}{n!(n+2)(n+1)}\right)$$

$$= 2^{n+1}\left(\frac{2(n+1)a_1-4na_0-na_1+2(n-1)a_0}{n!(n+2)(n+1)}\right) = 2^{n+1}\left(\frac{(n+2)a_1-2(n+1)a_0)}{(n+2)!}\right)$$

Thus $f(x) = \Sigma_{n=0}^{\infty}\frac{2^{n-1}(na_1-2(n-1)a_0)}{n!}x^n$

$$= a_1\Sigma_{n=0}^{\infty}\frac{2^{n-1}(n)}{n!}x^n - 2a_0\Sigma_{n=0}^{\infty}\frac{2^{n-1}(n-1)}{n!}x^n$$

$$= a_0(-2)\Sigma_{n=0}^{\infty}\frac{2^{n-1}(n-1)}{n!}x^n + a_1\Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^n$$

if these two series converge.

For what values of x does $\Sigma_{n=0}^{\infty}\frac{(n-1)2^{n-1}}{n!}x^n$ converge

Ratio test: Suppose we have the series Σb_n . Let $L = lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right|$

Then, if $L<1$, the series is absolutely convergent (and hence convergent).

If $L>1$, the series is divergent.

If $L=1$, the series may be divergent, conditionally convergent, or absolutely convergent.

$$lim_{n\to\infty}\left|\frac{\frac{n2^n}{(n+1)!}x^{n+1}}{\frac{(n-1)2^{n-1}}{n!}x^n}\right| = lim_{n\to\infty}\left|\frac{2nx}{(n+1)(n-1)}\right|$$

$$= 2x\, lim_{n\to\infty}\left|\frac{n}{(n+1)(n-1)}\right| = 0$$

Hence the series converges for all x

For what values of x does $\Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^n$ converge

$$lim_{n\to\infty}\left|\frac{\frac{2^n}{n!}x^{n+1}}{\frac{2^{n-1}}{(n-1)!}x^n}\right| = lim_{n\to\infty}\left|\frac{2x}{n}\right| = 2x\, lim_{n\to\infty}\left|\frac{1}{n}\right| = 0$$

Hence the series converges for all x

Thus the solution is

$$f(x) = a_0(-2)\Sigma_{n=0}^{\infty}\frac{2^{n-1}(n-1)}{n!}x^n + a_1\Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^n$$

and the domain is all real numbers.

I.e., the general solution is $f(x) = a_0\phi_0(x) + a_1\phi_1(x)$

where $\phi_0(x) = (-2)\Sigma_{n=0}^{\infty}\frac{2^{n-1}(n-1)}{n!}x^n$ and $\phi_1(x) = \Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^n$

Note we could have replaced the constant a_0 with $-2a_0$, but the a_i's have meaning: $a_n = \frac{f^{(n)}(0)}{n!}$. Thus our initial values are $a_0 = f(0)$ and $a_1 = f'(0)$