Second order differential equation:

Linear equation with constant coefficients:
If the second order differential equation is

$$ay'' + by' + cy = 0,$$

then $y = e^{rt}$ is a solution

Need to have two independent solutions.

Solve the following IVPs:

1.) $y'' - 6y' + 9y = 0$ $\quad$ $y(0) = 1,\ y'(0) = 2$

2.) $4y'' - y' + 2y = 0$ $\quad$ $y(0) = 3,\ y'(0) = 4$

3.) $4y'' + 4y' + y = 0$ $\quad$ $y(0) = 6,\ y'(0) = 7$

4.) $2y'' - 2y = 0$ $\quad$ $y(0) = 5,\ y'(0) = 9$

$ay'' + by' + cy = 0, \quad y = e^{rt}$, then
$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $r_1 \neq r_2$, then $b^2 - 4ac \neq 0$. Hence a general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$

If $b^2 - 4ac > 0$, general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$.

If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula.

general solution is $y = c_1e^{dt}cos(nt) + c_2e^{dt}sin(nt)$ where $r = d \pm in$

If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: te^{r_1t}

Hence general solution is $y = c_1e^{r_1t} + c_2te^{r_1t}$.

Initial value problem: use $y(t_0) = y_0$, $y'(t_0) = y'_0$ to solve for c_1, c_2 to find unique solution.

Derivation of general solutions:

If $b^2 - 4ac > 0$ we guessed e^{rt} is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

Section 3.3: If $b^2 - 4ac < 0$, :

Changed format of $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula:

$$e^{it} = cos(t) + isin(t)$$

Hence $e^{(d+in)t} = e^{dt}e^{int} = e^{dt}[cos(nt) + isin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$
$$= c_1 e^{dt}[cos(nt)+isin(nt)]+c_2 e^{dt}[cos(-nt)+isin(-nt)]$$
$$= c_1 e^{dt}cos(nt)+ic_1 e^{dt}sin(nt)+c_2 e^{dt}cos(nt)-ic_2 e^{dt}sin(nt)$$
$$=(c_1 + c_2)e^{dt}cos(nt) + i(c_1 - c_2)e^{dt}sin(nt)$$
$$= k_1 e^{dt}cos(nt) + k_2 e^{dt}sin(nt)$$

Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$.
Hence one solution is $y = e^{r_1 t}$ Need second solution.

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.

How about $y = v(t)e^{rt}$?

$$y' = v'(t)e^{rt} + v(t)re^{rt}$$
$$y'' = v''(t)e^{rt} + v'(t)re^{rt} + v'(t)re^{rt} + v(t)r^2e^{rt}$$
$$= v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2e^{rt}$$

$$ay'' + by' + cy = 0$$

$$a(v''e^{rt}+2v're^{rt}+vr^2e^{rt})+b(v'e^{rt}+vre^{rt})+cve^{rt} = 0$$

$$a(v''(t)+2v'(t)r+v(t)r^2)+b(v'(t)+v(t)r)+cv(t) = 0$$

$$av''(t)+2av'(t)r+av(t)r^2+bv'(t)+bv(t)r+cv(t) = 0$$

$$av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0$$

$$av''(t) + (2a(\tfrac{-b}{2a}) + b)v'(t) + 0 = 0$$

since $ar^2 + br + c = 0$ and $r = \frac{-b}{2a}$

$av''(t) + (-b + b)v'(t) = 0.$ Thus $av''(t) = 0.$

Hence $v''(t) = 0$ and $v'(t) = k_1$ and $v(t) = k_1 t + k_2$

Hence $v(t)e^{r_1 t} = (k_1 t + k_2)e^{r_1 t}$ is a soln

Thus $te^{r_1 t}$ is a nice second solution.

Hence general solution is $y = c_1 e^{r_1 t} + c_2 te^{r_1 t}$

3.5

homog

Thm: Suppose $c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to

$$ay'' + by' + cy = 0, \leftarrow \text{homog}$$

non-homog soln

If ψ is a solution to

$$ay'' + by' + cy = g(t) \; [*], \leftarrow \text{non homog } g(t) \neq 0$$

just need 1 soln

Then $\boxed{\psi + c_1\phi_1(t) + c_2\phi_2(t)}$ is also a solution to [*]. ys

Moreover if γ is also a solution to [*], then there exist constants c_1, c_2 such that

$$\gamma = \psi + c_1\phi_1(t) + c_2\phi_2(t)$$

a nonhomog + homog

Or in other words, $\psi + c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to [*].

Proof:

Define $\boxed{L(f) = af'' + bf' + cf.}$

Recall L is a linear function.

Let $h = c_1\phi_1(t) + c_2\phi_2(t)$. Since h is a solution to the differential equation, $ay'' + by' + cy = 0$,

homog

Since ψ is a solution to $ay'' + by' + cy = g(t)$, (*)

$$L(\psi) = a\psi'' + b\psi' + c\psi = g(t)$$

We will now show that $\psi + c_1\phi_1(t) + c_2\phi_2(t) = \psi + h$ is also a solution to [*].

$$L(\psi + h) = L(\psi) + L(h) = g(t) + 0 = g(t)$$

~~Since γ a solution to $ay'' + by' + cy = g(t)$,~~

$$L(\psi+h) = a(\psi+h)'' + b(\psi+h)' + c(\psi+h) = g(t)$$

$\Rightarrow \psi + h$ is a soln to (*)

~~We will first show that $\gamma - \psi$ is a solution to the differential equation $ay'' + by' + cy = 0$.~~

$$L(h) = L(c_1\phi_1 + c_2\phi_2) = c_1 L(\phi_1) + c_2 L(\phi_2) = c_1(0) + c_2(0) = 0$$

Since $\gamma - \psi$ is a solution to $ay'' + by' + cy = 0$ and

$c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to
$$ay'' + by' + cy = 0,$$

there exist constants c_1, c_2 such that

$\gamma - \psi =$ ____________________

Thus $\gamma = \psi + c_1\phi_1(t) + c_2\phi_2(t)$.

Thm:
Suppose f_1 is a a solution to $ay'' + by' + cy = g_1(t)$ and f_2 is a a solution to $ay'' + by' + cy = g_2(t)$, then $f_1 + f_2$ is a solution to $ay'' + by' + cy = g_1(t) + g_2(t)$

Proof:

Since f_1 is a solution to $ay'' + by' + cy = g_1(t)$,

Since f_2 is a solution to $ay'' + by' + cy = g_2(t)$,

We will now show that $f_1 + f_2$ is a solution to $ay'' + by' + cy = g_1(t) + g_2(t)$.

Sidenote: The proofs above work even if a, b, c are functions of t instead of constants.

Examples:

Find a suitable form for ψ for the following differential equations:

1.) $y'' - 4y' - 5y = 4e^{2t}$

Guess: $y = Ae^{2t}$

2.) $y'' - 4y' - 5y = 4sin(3t)$

Guess: $y = A\sin(3t) + B\cos(3t)$

3.) $y'' - 4y' - 5y = t^2 - 2t + 1$

$y = At^2 + Bt + C$

4.) $y'' - 5y = 4sin(3t)$

$y = A\sin(3t)$

5.) $y'' - 4y' = t^2 - 2t + 1$

$y = At^3 + Bt^2 + \ldots$

6.) $y'' - 4y' - 5y = 4(t^2 - 2t - 1)e^{2t}$

7.) $y'' - 4y' - 5y = 4sin(3t)e^{2t}$

8.) $y'' - 4y' - 5y = 4(t^2 - 2t - 1)sin(3t)e^{2t}$

9.) $y'' - 4y' - 5y = 4sin(3t) + 4sin(3t)e^{2t}$

10.) $y'' - 4y' - 5y$
$= 4sin(3t)e^{2t} + 4(t^2 - 2t - 1)e^{2t} + t^2 - 2t - 1$

11.) $\boxed{y'' - 4y' - 5y = 4sin(3t)} + 5cos(3t)$

12.) $y'' - 4y' - 5y = 4e^{-t}$

$y = te^{-t}$

To solve $ay'' + by' + cy = g_1(t) + g_2(t) + \ldots g_n(t)$ [**]

1.) Find the general solution to $ay'' + by' + cy = 0$:

$y = c_1\phi_1 + c_2\phi_2$ homog

2.) For each g_i, find a solution to $ay'' + by' + cy = g_i$:

$y = \psi_i \leftarrow$ one non homog for each g_i

This includes plugging guessed solution into $ay'' + by' + cy = g_i$ to find constant(s).

The general solution to [**] is

$y = c_1\phi_1 + c_2\phi_2 + \psi_1 + \psi_2 + \ldots\psi_n$

3.) If initial value problem:

Once general solution is known, can solve initial value problem (i.e., use initial conditions to find c_1, c_2).