Ch 3

Second order differential equation:

Linear equation with constant coefficients:

If the second order differential equation is

$$ay'' + by' + cy = 0,$$

then $y = e^{rt}$ is a solution

Need to have two independent solutions.

Solve the following IVPs:

1.) $y'' - 6y' + 9y = 0$ $\qquad y(0) = 1,\ y'(0) = 2$

2.) $4y'' - y' + 2y = 0$ $\qquad y(0) = 3,\ y'(0) = 4$

3.) $4y'' + 4y' + y = 0$ $\qquad y(0) = 6,\ y'(0) = 7$

4.) $2y'' - 2y = 0$ $\qquad y(0) = 5,\ y'(0) = 9$

$\boxed{ay'' + by' + cy = 0}$, $\boxed{y = e^{rt}}$, then

$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$, characteristic eq'n

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2 different real solns

If $r_1 \neq r_2$, then $b^2 - 4ac \neq 0$. Hence a general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$ 2 real soln to

real #'s

3.1 If $b^2 - 4ac > 0$, general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$.

3.3 2 different complex solns

If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula.

$y = c_1 e^{(d+in)t} + c_2 e^{(d-in)t}$ ugly unacceptable

general solution is $y = c_1e^{dt}cos(nt) + c_2e^{dt}sin(nt)$ where $r = d \pm in$

beautiful acceptable soln

If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: te^{r_1t}

3.4: 1 real soln: e^{rt}

Hence general solution is $y = c_1e^{r_1t} + c_2te^{r_1t}$.

Initial value problem: use $y(t_0) = y_0$, $y'(t_0) = y'_0$ to solve for c_1, c_2 to find unique solution.

Ch 3

Derivation of general solutions:

3.1

If $b^2 - 4ac > 0$ we guessed e^{rt} is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

Section 3.3: If $b^2 - 4ac < 0$, :

ugly when r_1, r_2 are complex

Changed format of $y = c_1e^{r_1t} + c_2e^{r_2t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula:

$$\boxed{e^{it} = cos(t) + isin(t)}$$

Hence $e^{(d+in)t} = e^{dt}e^{int} = e^{dt}[cos(nt) + isin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$y = c_1e^{r_1t} + c_2e^{r_2t}$ ugly

$= c_1e^{dt}[cos(nt)+isin(nt)]+c_2e^{dt}[cos(-nt)+isin(-nt)]$

$= c_1e^{dt}cos(nt)+ic_1e^{dt}sin(nt)+c_2e^{dt}cos(nt)-ic_2e^{dt}sin(nt)$

$=(c_1 + c_2)e^{dt}cos(nt) + i(c_1 - c_2)e^{dt}sin(nt)$

$y = k_1e^{dt}cos(nt) + k_2e^{dt}sin(nt)$ acceptable

Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$.
Hence one solution is $y = e^{r_1t}$ Need second solution.

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.

How about $\boxed{y = v(t)e^{rt}?}$

$y' = v'(t)e^{rt} + v(t)re^{rt}$

$y'' = v''(t)e^{rt} + v'(t)re^{rt} + v'(t)re^{rt} + v(t)r^2e^{rt}$

$= v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2e^{rt}$

$ay'' + by' + cy = 0$

$a(v''e^{rt}+2v're^{rt}+vr^2e^{rt})+b(v'e^{rt}+vre^{rt})+cve^{rt} = 0$

$a(v''(t)+2v'(t)r+v(t)r^2)+b(v'(t)+v(t)r)+cv(t) = 0$

$av''(t)+2av'(t)r+av(t)r^2+bv'(t)+bv(t)r+cv(t) = 0$

$av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0$

$av''(t) + (2a(\frac{-b}{2a}) + b)v'(t) + 0 = 0$

since $ar^2 + br + c = 0$ and $r = \frac{-b}{2a}$ e^{rt} soln

$av''(t) + (-b + b)v'(t) = 0$. Thus $av''(t) = 0$.

Hence $v''(t) = 0$ and $v'(t) = k_1$ and $v(t) = k_1t + k_2$

Hence $v(t)e^{r_1t} = (k_1t + k_2)e^{r_1t}$ is a soln

Thus te^{r_1t} is a nice second solution.

Hence general solution is $\boxed{y = c_1e^{r_1t} + c_2te^{r_1t}}$

one repeated root