Is $f(x) = 2^x$ linear?

$f(0+0) = \cancel{2^{0}}\ f(0) = 2^0 = 1$

$f(0)+f(0) = 2^0 + 2^0 = 1+1 = 2$

$\Big\}\ 1 \neq 2 \Rightarrow f$ is not linear

Linear Functions

A function f is linear if $f(a\mathbf{x} + b\mathbf{y}) = af(\mathbf{x}) + bf(\mathbf{y})$

Or equivalently f is linear if 1.) $f(a\mathbf{x}) = af(\mathbf{x})$ and 2.) $f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$

Theorem: If f is linear, then $f(\mathbf{0}) = \mathbf{0}$

Proof: $f(\mathbf{0}) = f(0 \cdot \mathbf{0}) = 0 \cdot f(\mathbf{0}) = \mathbf{0}$

Example 1a.) $f : R \to R$, $f(x) = 2x$

Proof:
$f(ax+by) = 2(ax+by) = 2ax + 2by = af(x) + bf(y)$

Example 1b.) $f : R \to R$, $f(x) = 2x + 3$ is NOT linear.

Proof: $f(2 \cdot 0) = f(0) = 3$, but $2f(0) = 2 \cdot 3 = 6$.
Hence $f(2 \cdot 0) \neq 2f(0)$

Alternate Proof: $f(0+1) = f(1) = 5$, but $f(0)+f(1) = 3+5 = 8$. Hence $f(0+1) \neq f(0)+f(1)$

Note confusing notation: Most lines, $f(x) = mx + b$ are not linear functions.

Question: When is a line, $f(x) = mx + b$, a linear function?

Example 2.) $f : R^2 \to R^2$,
$f((x_1, x_2)) = (2x_1, x_1 + x_2)$

Proof: Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$

$a\mathbf{x}+b\mathbf{y} = a(x_1, x_2) + b(y_1, y_2) = (ax_1, ax_2) + (by_1, by_2) = (ax_1 + by_1, ax_2 + by_2)$

$$
\begin{aligned}
f(ax_1 + by_1, ax_2 + by_2)\\
&= (2(ax_1 + by_1), ax_1 + by_1 + ax_2 + by_2)\\
&= (2ax_1 + 2by_1, ax_1 + ax_2 + by_1 + by_2)\\
&= (2ax_1, ax_1 + ax_2) + (2by_1, by_1 + by_2)\\
&= a(2x_1, x_1 + x_2) + b(2y_1, y_1 + y_2)\\
&= af((x_1, x_2)) + bf((y_1, y_2))
\end{aligned}
$$

Example 3.) D : set of all differential functions $\to$ set of all functions, $D(f) = f'$

Proof:
$D(af+bg) = (af+bg)' = af' + bg' = aD(f) + bD(g)$

$D(2x^3 + \sin x) = D(2x^3) + D(\sin x)$

$= 2D(x^3) + D(\sin x) = 2 \cdot 3x^2 + \cos x$

$= 6x^2 + \cos x$

Example 4.) Given a, b real numbers,
I : set of all integrable functions on [a, b] $\to R$,
$I(f) = \int_a^b f$

Proof: $I(sf + tg) = \int_a^b sf + tg = s\int_a^b f + t\int_a^b g = sI(f) + tI(g)$

Example 5.) The inverse of a linear function is linear (when the inverse exists).

Suppose $f^{-1}(x) = c$, $f^{-1}(y) = d$.

Then $f(c) = x$ and $f(d) = y$ and
$f(ac + bd) = af(c) + bf(d) = ax + by$.

Hence $f^{-1}(ax + by) = ac + bd = af^{-1}(x) + bf^{-1}(y)$.

Example 6.) D : set of all twice differential functions $\to$ set of all functions, $L(f) = af'' + bf' + cf$

Proof:

$$L(sf + tg) = a(sf + tg)'' + b(sf + tg)' + c(sf + tg)$$
$$= saf'' + tag'' + sbf' + tbg' + scf + tcg$$
$$= s(af'' + bf' + cf) + t(ag'' + bg' + cg)$$
$$= sL(f) + tL(g)$$

$af'' + bf' + cf = 0$

ψ_1 soln $\Longleftrightarrow L(\psi_1) = 0$

Consequence 1: If ψ_1, ψ_2 are solutions to $af'' + bf' + cf = 0$, then $3\psi_1 + 5\psi_2$ is also a solution to $af'' + bf' + cf = 0$,

Proof: Since ψ_1, ψ_2 are solutions to $af'' + bf' + cf = 0$, $L(\psi_1) = 0$ and $L(\psi_2) = 0$.

Hence $L(3\psi_1 + 5\psi_2) = 3L(\psi_1) + 5L(\psi_2)$
$$= 3(0) + 5(0) = 0. \Rightarrow 3\psi_1 + 5\psi_2 \text{ is also a sol}$$

Thus $3\psi_1 + 5\psi_2$ is also a solution to $af'' + bf' + cf = 0$

Consequence 2:
If ψ_1 is a solution to $af'' + bf' + cf = h$
and ψ_2 is a solution to $af'' + bf' + cf = k$,
then $3\psi_1 + 5\psi_2$ is a solution to $af'' + bf' + cf = 3h + 5k$,

Since ψ_1 is a solution to $af'' + bf' + cf = h$, $L(\psi_1) = h$.

Since ψ_2 is a solution to $af'' + bf' + cf = k$, $L(\psi_2) = k$.

Hence $L(3\psi_1 + 5\psi_2) = 3L(\psi_1) + 5L(\psi_2)$
$$= 3h + 5k.$$

Thus $3\psi_1 + 5\psi_2$ is also a solution to
$$af'' + bf' + cf = 3h + 5k$$