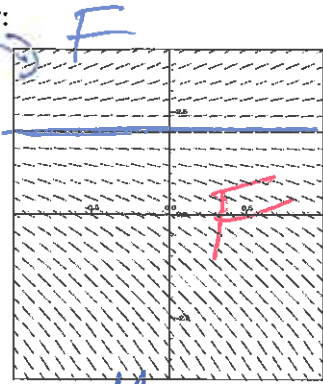$y' = f(y)$ autonomous

2.) Circle the differential equation whose direction field is given below:

A) $y' = t^2$

B) $y' = \frac{1}{2}$

C) $y' = 1$

D) $y' = -1$

E) $y' = y + 1$

F) $y' = y - 2$

G) $y' = (y+1)(y-2)$

H) $y' = (y+1)^2(y-2)^2$

I) $y' = (y+1)(y-2)^2$

J) $y' = (y+1)^2(y-2)$

K) $y' = -\frac{t}{y}$

F

$y = 2$

F

semi-stably equilibrium

$(+)$

$(-)$

stable equilibrium

G

$(+)$

$(+)$

I

$(-)$

H

$(+)$

$(+)$

$(+)$

K

$y' = \frac{t}{y}$

A
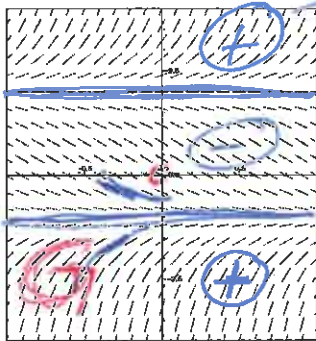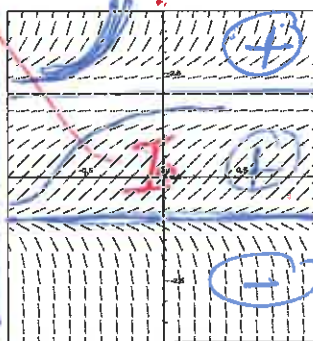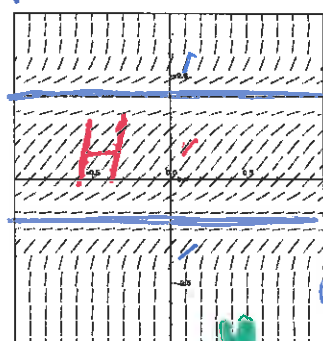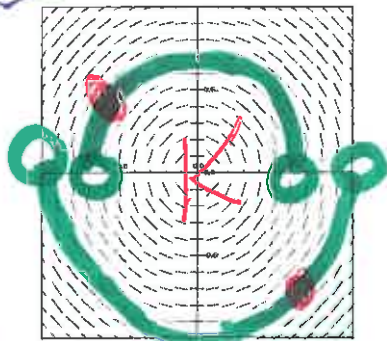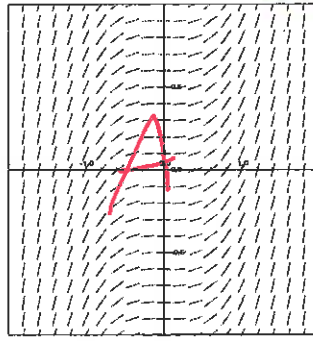
$y' = t^2$

non-autonomous

$(+)$
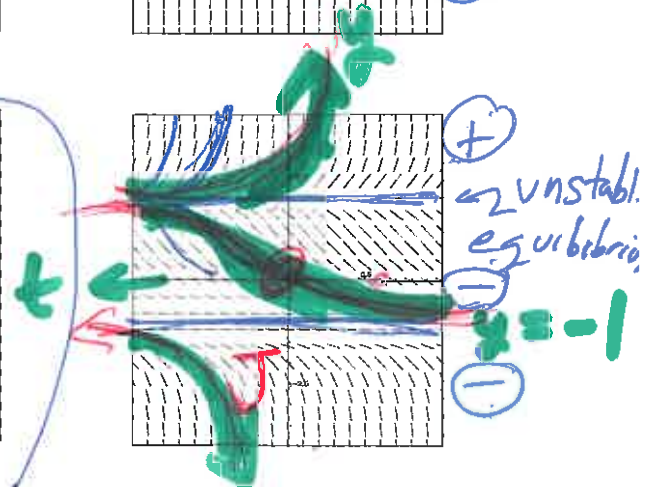
unstable equilibrium

$y = -1$

$(-)$

$\xleftarrow{\quad}$ $\;\;$ $\xrightarrow{\quad}$

$-1 \quad\quad 2$

Domain for IVP
possible t values satisfy
IVP including ODE

Long term
behaviour
(normally depends
on initial
condition value

Second order differential equation:

Linear equation with constant coefficients:

If the second order differential equation is

$$ay'' + by' + cy = 0,$$

then $y = e^{rt}$ is a solution

Need to have two independent solutions.

Solve the following IVPs:

1.) $y'' - 6y' + 9y = 0$ $y(0) = 1,\ y'(0) = 2$

2.) $4y'' - y' + 2y = 0$ $y(0) = 3,\ y'(0) = 4$

3.) $4y'' + 4y' + y = 0$ $y(0) = 6,\ y'(0) = 7$

4.) $2y'' - 2y = 0$ $y(0) = 5,\ y'(0) = 9$

$ay'' + by' + cy = 0, \quad y = e^{rt}$, then

$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $r_1 \neq r_2$, then $b^2 - 4ac \neq 0$. Hence a general solution is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

If $b^2 - 4ac > 0$, general solution is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula.

general solution is $y = c_1 e^{dt} \cos(nt) + c_2 e^{dt} \sin(nt)$ where $r = d \pm in$

If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: $te^{r_1 t}$

Hence general solution is $y = c_1 e^{r_1 t} + c_2 te^{r_1 t}$.

Initial value problem: use $y(t_0) = y_0$, $y'(t_0) = y_0'$ to solve for c_1, c_2 to find unique solution.

Derivation of general solutions:

3.

If $b^2 - 4ac > 0$ we guessed e^{rt} is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

Section 3.3: If $b^2 - 4ac < 0,$:

Changed format of $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula:
$$e^{it} = cos(t) + i sin(t)$$

Hence $e^{(d+in)t} = e^{dt} e^{int} = e^{dt}[cos(nt) + i sin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
$= c_1 e^{dt}[cos(nt)+i sin(nt)]+c_2 e^{dt}[cos(-nt)+i sin(-nt)]$
$= c_1 e^{dt} cos(nt)+ic_1 e^{dt} sin(nt)+c_2 e^{dt} cos(nt)-ic_2 e^{dt} sin(nt)$
$=(c_1 + c_2)e^{dt} cos(nt) + i(c_1 - c_2)e^{dt} sin(nt)$
$= k_1 e^{dt} cos(nt) + k_2 e^{dt} sin(nt)$

Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$.
Hence one solution is $y = e^{r_1 t}$ Need second solution.

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.
How about $y = v(t)e^{rt}$?

$y' = v'(t)e^{rt} + v(t)re^{rt}$

$y'' = v''(t)e^{rt} + v'(t)re^{rt} + v'(t)re^{rt} + v(t)r^2 e^{rt}$
$= v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2 e^{rt}$

$ay'' + by' + cy = 0$

$a(v''e^{rt}+2v're^{rt}+vr^2 e^{rt})+b(v'e^{rt}+vre^{rt})+cve^{rt} = 0$

$a(v''(t)+2v'(t)r+v(t)r^2)+b(v'(t)+v(t)r)+cv(t) = 0$

$av''(t)+2av'(t)r+av(t)r^2+bv'(t)+bv(t)r+cv(t) = 0$

$av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0$

$av''(t) + (2a(\frac{-b}{2a}) + b)v'(t) + 0 = 0$
$\qquad$ since $ar^2 + br + c = 0$ and $r = \frac{-b}{2a}$

$av''(t) + (-b + b)v'(t) = 0.$ $\qquad$ Thus $av''(t) = 0.$

Hence $v''(t) = 0$ and $v'(t) = k_1$ and $v(t) = k_1 t + k_2$

$\qquad$ Hence $v(t)e^{r_1 t} = (k_1 t + k_2)e^{r_1 t}$ is a soln

Thus $te^{r_1 t}$ is a nice second solution.

Hence general solution is $y = c_1 e^{r_1 t} + c_2 te^{r_1 t}$