4.) Circle the general solution to the differential equation whose direction field is given below:

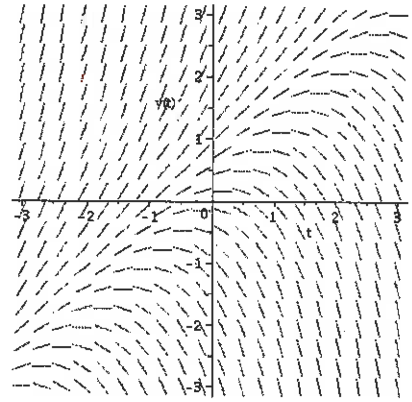
A) $y = t + C$ B) $y = t^2 + C$

C) $y = e^t + C$ D) $y = Ce^t + t + 1$

E) $y = Ce^t$ F) $y = e^t + t + C$

G) $y = ln(t) + C$ H) $y = C$

I) $y = sin(t) + C$ J) $y = cos(t) + C$



5.) Which of the following could be the general solution to the differential equation whose direction field is given below:

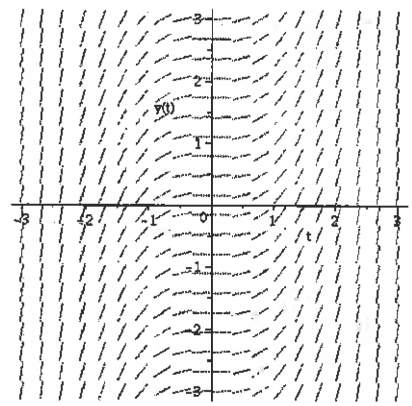
~~A) $y = t + C$~~ ~~B) $y = t^2 + C$~~

~~C) $y = e^t + C$~~ D) $y = \frac{(t-1)^3}{3} + C$

~~E) $y = Ce^t$~~ F) $y = \frac{t^3}{3} + C$

~~G) $y = ln(t) + C$~~ ~~H) $y = C$~~

~~I) $y = \frac{Ct^3}{3}$~~ ~~J) $y = \frac{C(t-1)^3}{3}$~~



6.) Circle the differential equation whose direction field is given below:

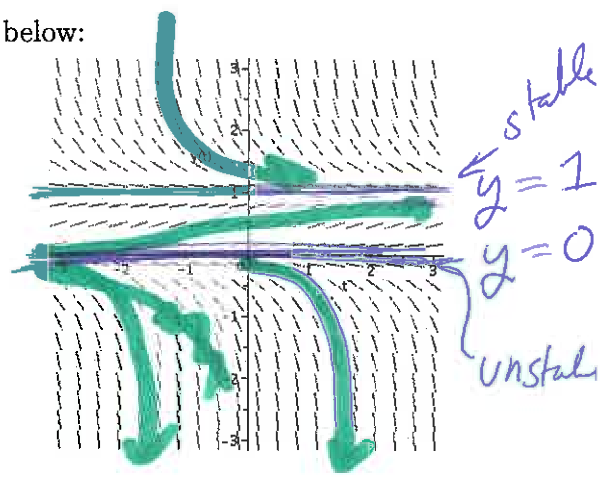
~~A) $y' = t^2$~~ B) $y' = y + 3$

~~C) $y' = e^t$~~ ~~D) $y' = t + 1$~~

~~E) $y' = t - y$~~ ~~F) $y' = y - t$~~

G) $y' = (1 + y)(1 - y)$ H) $y' = y(1 + y)$

~~I) $y' = t(1 - t)$~~ J) $y' = y(1 - y)$



Equilibrium soln

$y = 0$: asymptotically ~~stable~~ unstable

$y = 1$: asymptotically stable

Linear algebra pre-requisites you must know.

$\mathbf{b_1}, ..., \mathbf{b_n}$ are linearly independent if

$c_1\mathbf{b_1} + c_2\mathbf{b_2} + ... + c_n\mathbf{b_n} = d_1\mathbf{b_1} + d_2\mathbf{b_2} + ... + d_n\mathbf{b_n}$ implies $c_1 = d_1,\ c_2 = d_2 ...,\ c_n = d_n$.

or equivalently,

$\mathbf{b_1}, ..., \mathbf{b_n}$ are linearly independent if

$c_1\mathbf{b_1} + c_2\mathbf{b_2} + ... + c_n\mathbf{b_n} = 0$ implies $c_1 = c_2 = ... c_n$.

Example 1: $\mathbf{b_1} = (1,0,0),\ \mathbf{b_2} = (0,1,0),\ \mathbf{b_3} = (0,0,1)$.■

$$(1,2,3) \neq (1,2,4).$$

If $(a,b,c) = (1,2,3)$ then $a = 1,\ b = 2,\ c = 3$.

Example 2: $\mathbf{b_1} = 1,\ \mathbf{b_2} = t,\ \mathbf{b_3} = t^2$.

$$1 + 2t + 3t^2 \neq 1 + 2t + 4t^2.$$

If $a + bt + ct^2 = 1 + 2t + 3t^2$ then $a = 1,\ b = 2,\ c = 3$.

Application: Partial Fractions

$(x^2+1)(x-3)$

$$\frac{4}{(x^2+1)(x-3)} = \left(\frac{Ax+B}{x^2+1} + \frac{C}{x-3}\right)(x^2+1)(x-3)$$

$$= \frac{(Ax+B)(x-3)+C(x^2+1)}{(x^2+1)(x-3)}$$

Hence $\frac{4}{(x^2+1)(x-3)} = \frac{(Ax+B)(x-3)+C(x^2+1)}{(x^2+1)(x-3)}$

Thus $4 = (Ax+B)(x-3) + C(x^2+1)$

$$4 = Ax^2 + Bx - 3Ax - 3B + Cx^2 + C$$

$$4 = (A+C)x^2 + (B-3A)x - 3B + C$$

I.e., $0x^2+0x+4 = (A+C)x^2 + (B-3A)x - 3B + C$

Thus $0 = A + C,\ 0 = B - 3A,\ 4 = -3B + C$.

$C = -A,\ B = 3A,$
$4 = -3(3A) + -A$ implies $4 = -10A$.
Hence $A = -\frac{2}{5},\ B = 3(-\frac{2}{5}) = -\frac{6}{5},\ C = \frac{2}{5}$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & -3 & 1 & 4 \end{array}\right]$$

Thus, $\frac{4}{(x^2+1)(x-3)} = \frac{-\frac{2}{5}x - \frac{6}{5}}{x^2+1} + \frac{\frac{2}{5}}{x-3}$

$$= \frac{-2x-6}{5(x^2+1)} + \frac{2}{5(x-3)}$$

linear independence $\Longleftrightarrow$ unique representations. when representation exists

Suppose salty water enters and leaves a tank at a rate of 2 liters/minute.

Suppose also that the salt concentration of the water entering the tank varies with respect to time according to $Q(t) \cdot tsin(t^2)$ g/liters where $Q(t)$ = amount of salt in tank in grams. (Note: this is not realistic).

If the tank contains 4 liters of water and initially contains 5g of salt, find a formula for the amount of salt in the tank after t minutes.

Let $Q(t)$ = amount of salt in tank in grams.

Note $Q(0) = 5$ g

rate in = (2 liters/min)($Q(t) \cdot tsin(t^2)$ g/liters)

$= 2Qtsin(t^2)$ g/min

rate out = (2 liters/min)$(\frac{Q(t)g}{4\text{liters}}) = \frac{Q}{2}$ g/min

$\frac{dQ}{dt}$ = rate in - rate out $= 2Qtsin(t^2) - \frac{Q}{2}$

$$\boxed{\frac{dQ}{dt} = Q(2tsin(t^2) - \frac{1}{2})}$$

This is a first order linear ODE. It is also a separable ODE. Thus can use either 2.1 or 2.2 methods.

Using the easier 2.2:

$$\int \frac{dQ}{Q} = \int (2tsin(t^2) - \frac{1}{2})dt = \int 2tsin(t^2)dt - \int \frac{1}{2}dt$$

Let $u = t^2$, $du = 2tdt$

$$ln|Q| = \int sin(u)du - \frac{t}{2} = -cos(u) - \frac{t}{2} + C$$
$$= -cos(t^2) - \frac{t}{2} + C$$

$$|Q| = e^{-cos(t^2) - \frac{t}{2} + C} = e^C e^{-cos(t^2) - \frac{t}{2}}$$

$$Q = Ce^{-cos(t^2) - \frac{t}{2}}$$

$Q(0) = 5: \quad 5 = Ce^{-1-0} = Ce^{-1}$. Thus $C = 5e$

Thus $Q(t) = 5e \cdot e^{-cos(t^2) - \frac{t}{2}}$

Thus $Q(t) = 5e^{-cos(t^2) - \frac{t}{2} + 1}$

Long-term behaviour:

$Q(t) = 5(e^{-cos(t^2)})(e^{\frac{-t}{2}})e$

As $t \to \infty$, $e^{\frac{-t}{2}} \to 0$, while $5(e^{-cos(t^2)})e$ are finite.

Thus as $t \to \infty$, $Q(t) \to 0$.

Calulus pre-requisites you must know.

Derivative = slope of tangent line = rate.

Integral = area between curve and x-axis (where area can be negative).

The Fundamental Theorem of Calculus: Suppose f continuous on $[a, b]$.

1.) If $G(x) = \int_a^x f(t)dt$, then $G'(x) = f(x)$.

I.e., $\frac{d}{dx}[\int_a^x f(t)dt] = f(x)$.

2.) $\int_a^b f(t)dt = F(b) - F(a)$ where F is any antiderivative of f, that is $F' = f$.

Suppose f is cont. on (a, b) and the point $t_0 \in (a, b)$,
Solve IVP: $\frac{dy}{dt} = f(t)$, $y(t_0) = y_0$ ← Calc 1

$$dy = f(t)dt$$

$$\int dy = \int f(t)dt$$

$y = F(t) + C$ where F is any anti-derivative of F.

Initial Value Problem (IVP): $y(t_0) = y_0$

$y_0 = F(t_0) + C$ implies $C = y_0 - F(t_0)$ ← Find unique C

Hence unique solution (if domain connected) to IVP:

$$y = F(t) + y_0 - F(t_0)$$

CH 2: Solve $\frac{dy}{dt} = f(t, y)$

1.1: Direction Fields ********************

*******Existence/Uniqueness of solution**********

Thm 2.4.2: Suppose the functions $z = f(t, y)$ and $z = \frac{\partial f}{\partial y}(t, y)$ are cont. on $(a, b) \times (c, d)$ and the point $(t_0, y_0) \in (a, b) \times (c, d)$, then there exists an interval $(t_0 - h, t_0 + h) \subset (a, b)$ such that there exists a unique function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

$$y' = f(t, y), \quad y(t_0) = y_0.$$

Thm 2.4.1: If p and g are continuous on (a, b) and the point $t_0 \in (a, b)$, then there exists a unique function $y = \phi(t)$ defined on (a, b) that satisfies the following initial value problem:

$$\boxed{y' + p(t)y = g(t), \quad y(t_0) = y_0.}$$

But in general, $y' = f(t, y)$, solution may or may not exist and solution may or may not be unique.

Linear vs Non-linear

linear: $a_0(t)y^{(n)} + ... + a_n(t)y = g(t)$

Determine if linear or non-linear:

Ex: $ty'' - t^3y' - 3y = sin(t)$

Ex: $2y'' - 3y' - 3y^2 = 0$

********Existence of a solution**************

********Uniqueness of solution***************

CH 2: Solve $\frac{dy}{dt} = f(t, y)$

2.2: Separation of variables: $N(y)dy = P(t)dt$

2.1: First order linear eqn: $\frac{dy}{dt} + p(t)y = g(t)$

Ex 1: $t^2y' + 2ty = tsin(t)$

Ex 2: $y' = ay + b$

Ex 3: $y' + 3t^2y = t^2$, $y(0) = 0$

Note: could use section 2.2 method, separation of variables to solve ex 2 and 3.

Ex 1: $t^2y' + 2ty = sin(t)$
(note, cannot use separation of variables).

$t^2y' + 2ty = sin(t)$

$(t^2y)' = sin(t)$

$\int (t^2y)'dt = \int sin(t)dt$

$(t^2y) = -cos(t) + C$ implies $y = -t^{-2}cos(t) + Ct^{-2}$

Gen ex: Solve $y' + p(x)y = g(x)$

Let $F(x)$ be an anti-derivative of $p(x)$

$e^{F(x)}y' + [p(x)e^{F(x)}]y = g(x)e^{F(x)}$

$e^{F(x)}y' + [F'(x)e^{F(x)}]y = g(x)e^{F(x)}$

$[e^{F(x)}y]' = g(x)e^{F(x)}$

$e^{F(x)}y = \int g(x)e^{F(x)}dx$

$y = e^{-F(x)} \int g(x)e^{F(x)}dx$

$y = e^{-F(x)}A(x) + Ce^{-F(x)}$

unique soln exists for C given initial value

Sol'n does not exist to IVP

Ex 1: $y' = y' + 1 \Rightarrow 0 = 1 \Rightarrow$ no sol'n

Ex 2: $(y')^2 = -1 \Rightarrow$ no real value function is a sol'n

IVP ex 3: $\frac{dy}{dx} = y(1 + \frac{1}{x})$, $y(0) = 1$ $\quad x \neq 0$

$\int \frac{dy}{y} = \int (1 + \frac{1}{x})dx \quad$ implies $\quad ln|y| = x + ln|x| + C$

$|y| = e^{x+ln|x|+C} = e^x e^{ln|x|} e^C = C|x|e^x = Cxe^x$

$y = \pm Cxe^x$ implies $y = Cxe^x$

$y(0) = 1$: $\quad 1 = C(0)e^0 = 0$ implies

IVP $\frac{dy}{dx} = y(1 + \frac{1}{x})$, $y(0) = 1$ has no solution.

See direction field created using
www.math.rutgers.edu/∼ sontag/JODE/JOdeApplet.html

Ex Non-unique: $y' = y^{\frac{1}{3}}$

$y = 0$ is a solution to $y' = y^{\frac{1}{3}}$ since $y' = 0 = 0^{\frac{1}{3}} = y^{\frac{1}{3}}$

Suppose $y \neq 0$. Then $\frac{dy}{dx} = y^{\frac{1}{3}}$ implies $y^{-\frac{1}{3}}dy = dx$

$\int y^{-\frac{1}{3}}dy = \int dx$ implies $\frac{3}{2}y^{\frac{2}{3}} = x + C$

$y^{\frac{2}{3}} = \frac{2}{3}x + C$ implies $y = \pm\sqrt{(\frac{2}{3}x + C)^3}$

Suppose $y(3) = 0$. Then $0 = \sqrt{(2 + C)^3}$ implies $C = -2$. $\blacksquare$

Thus initial value problem, $y' = y^{\frac{1}{3}}$, $y(3) = 0$, has 3 sol'ns: $\blacksquare$

$$y = 0, \quad y = \sqrt{(\tfrac{2}{3}x - 2)^3}, \quad y = -\sqrt{(\tfrac{2}{3}x - 2)^3}$$

2.4 #27b. Solve Bernoulli's equation,

$$y' + p(t)y = g(t)y^n,$$

when $n \neq 0, 1$ by changing it

$$y^{-n}y' + p(t)y^{1-n} = g(t)$$

when $n \neq 0, 1$ by changing it to a linear equation by substituting $v = y^{1-n}$

Solve $ty' + 2t^{-2}y = 2t^{-2}y^5$

Section 2.5: Solve $\frac{dy}{dt} = f(y)$

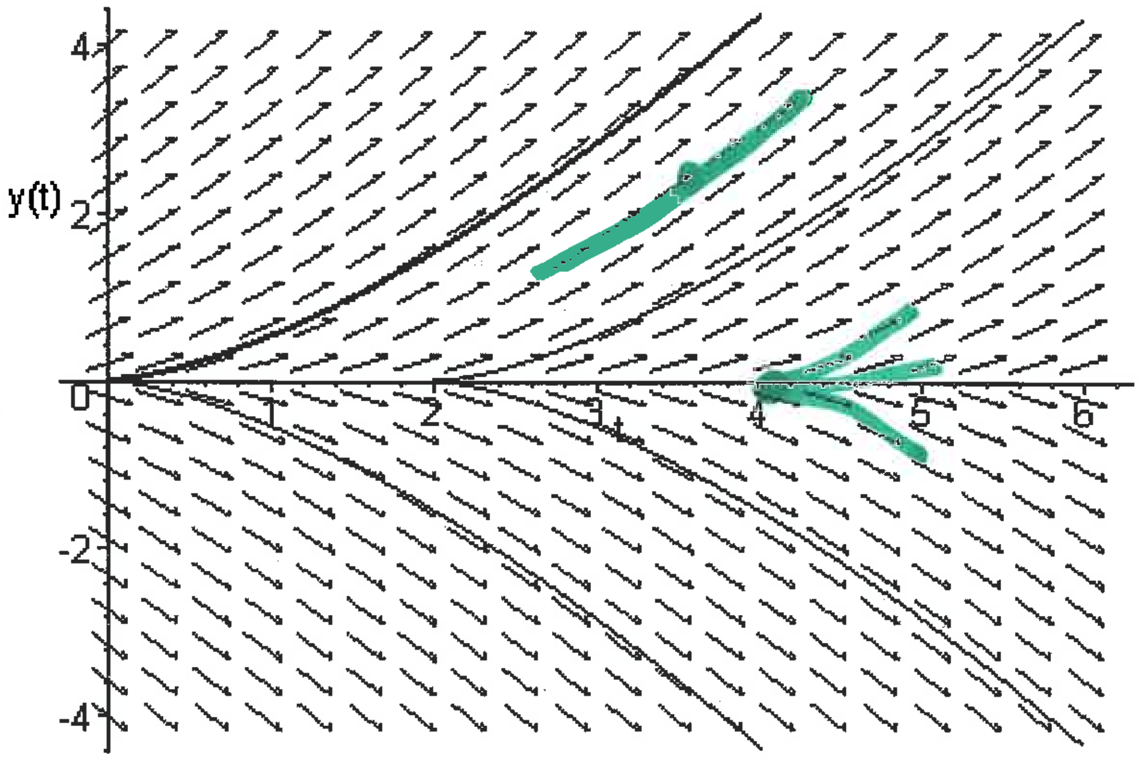
If given either differential equation $y' = f(y)$ OR direction field:

Find equilibrium solutions and determine if stable, unstable, semi-stable.

Understand what the above means.

http://homepage.math.uiowa.edu/~idarcy/COURSES/100/SPRING15/3600.html

$$y' = y^{1/3}$$



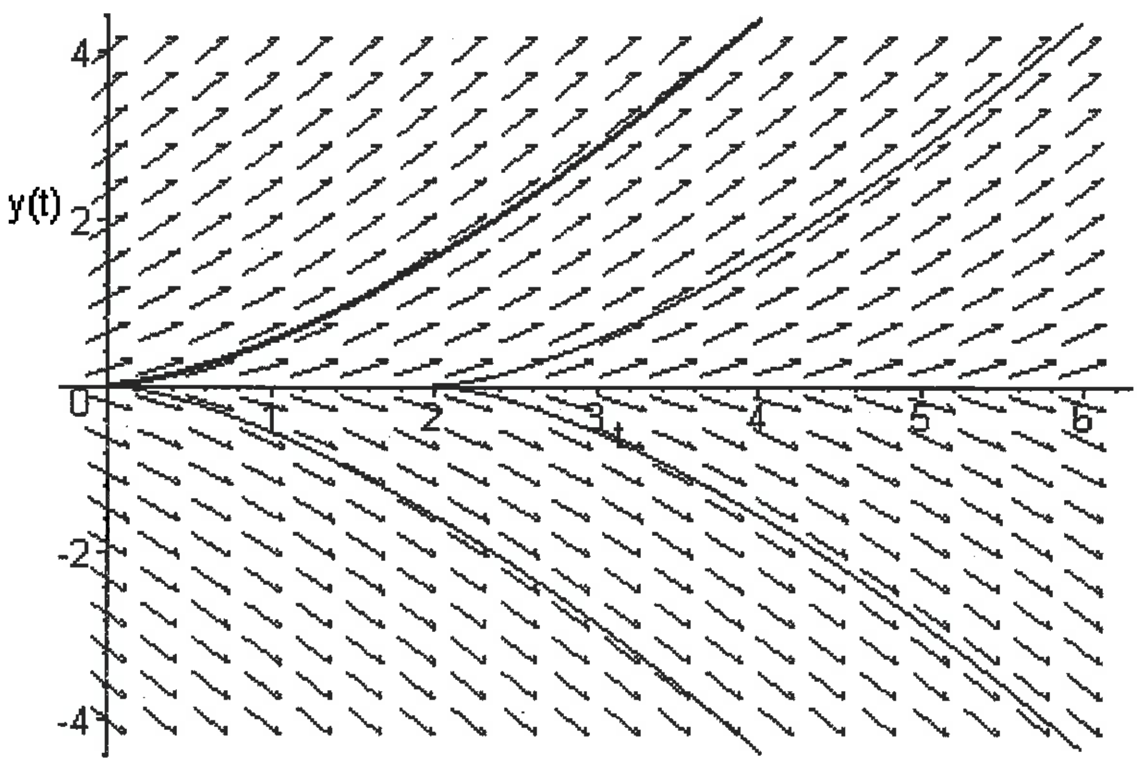


Figure 2.4.1 from ***Elementary Differential Equations and Boundary Value Problems*, Eighth Edition** by William E. Boyce and Richard C. DiPrima