Existence and Uniqueness

## 1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a, b) \to R$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$y' + p(t)y = g(t),$$
$$y(t_0) = y_0$$

## 2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a, b) \to R$, $q : (a, b) \to R$, and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$y'' + p(t)y' + q(t)y = g(t),$$
$$y(t_0) = y_0,$$
$$y'(t_0) = y'_0$$

$g(t) = 0$

Thm 3.2.2: If $\phi_1$ and $\phi_2$ are two solutions to a homogeneous linear differential equation, the $c_1\phi_1 + c_2\phi_2$ is also a solution to this linear differential equation.

If $\phi_i$ are sol'ns to linear homog DE

$$\Rightarrow \sum_{i=1}^{n} c_i \phi_i \text{ is also a sol'n}$$

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Need $n$ $\phi_i$ for $n^{th}$ order lin diff eqn

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$1^{rst}$ order: gen sol'n:

$$y = c\,\phi$$

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$2^{nd}$ order gen soln

$$y = c_1 \phi_1 + c_2 \phi_2$$

where $\phi_i$ are lin indep. per Thm 3.2.4 plus...

Pf of Thm 3.2.2

Lin $2^{nd}$ order Hom D. E.

homogenous

$$a y'' + b y' + c y = 0 \quad (\star)$$

Note: $a, b, c$ can be functions of $t$

$\phi_1$ is a soln to $(\star)$

$$\Rightarrow a\phi_1'' + b\phi_1' + c\phi_1 = 0$$

$\phi_2$ is a soln to $(\star)$

$$\Rightarrow a\phi_2'' + b\phi_2' + c\phi_2 = 0$$

$\boxed{\text{Claim: } c_1\phi_1 + c_2\phi_2 \text{ is a soln to } (\star)}$

$$a(c_1\phi_1 + c_2\phi_2)'' + b(c_1\phi_1 + c_2\phi_2)' + c(c_1\phi_1 + c_2\phi_2)$$

$$= \underline{ac_1\phi_1''} + ac_2\phi_2'' + \underline{bc_1\phi_1'} + bc_2\phi_2' + \underline{cc_1\phi_1} + cc_2\phi_2$$

$$= c_1(a\phi_1'' + b\phi_1' + c\phi_1) + c_2(a\phi_2'' + b\phi_2' + c\phi_2)$$

$$= c_1(0) + c_2(0) = 0 \Rightarrow c_1\phi_1 + c_2\phi_2$$

is a soln to $(\star)$

Linear Functions

A function $f$ is linear if $f(a\mathbf{x} + b\mathbf{y}) = af(\mathbf{x}) + bf(\mathbf{y})$

Or equivalently $f$ is linear if 1.) $f(a\mathbf{x}) = af(\mathbf{x})$ and 2.) $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$

Theorem: If $f$ is linear, then $f(\mathbf{0}) = \mathbf{0}$

Proof: $f(\mathbf{0}) = f(0 \cdot \mathbf{0}) = 0 \cdot f(\mathbf{0}) = \mathbf{0}$

Example 1a.) $f : R \to R$, $f(x) = 2x$

Proof:
$f(ax + by) = 2(ax + by) = 2ax + 2by = af(x) + bf(y)$

Example 1b.) $f : R \to R$, $f(x) = 2x + 3$ is NOT linear.

Proof: $f(2 \cdot 0) = f(0) = 3$, but $2f(0) = 2 \cdot 3 = 6$.
Hence $f(2 \cdot 0) \neq 2f(0)$

Alternate Proof: $f(0 + 1) = f(1) = 5$, but
$f(0) + f(1) = 3 + 5 = 8$. Hence $f(0+1) \neq f(0) + f(1)$

Note confusing notation: Most lines, $f(x) = mx + b$ are not linear functions.



Question: When is a line, $f(x) = mx + b$, a linear function?

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Example 2.) $f : R^2 \to R^2$,
$$f((x_1, x_2)) = (2x_1, x_1 + x_2)$$

Proof: Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$

$a\mathbf{x} + b\mathbf{y} = a(x_1, x_2) + b(y_1, y_2) = (ax_1, ax_2) + (by_1, by_2) = (ax_1 + by_1, ax_2 + by_2)$

$$\begin{aligned} f(ax_1 + by_1, ax_2 + by_2) &= (2(ax_1 + by_1), ax_1 + by_1 + ax_2 + by_2) \\ &= (2ax_1 + 2by_1, ax_1 + ax_2 + by_1 + by_2) \\ &= (2ax_1, ax_1 + ax_2) + (2by_1, by_1 + by_2) \\ &= a(2x_1, x_1 + x_2) + b(2y_1, y_1 + y_2) \\ &= af((x_1, x_2)) + bf((y_1, y_2)) \end{aligned}$$

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Example 3.) $D$ : set of all differential functions $\to$ set of all functions, $D(f) = f'$

Proof:
$D(af + bg) = (af + bg)' = af' + bg' = aD(f) + bD(g)$

Example 4.) Given $a, b$ real numbers,
$I$ : set of all integrable functions on [a, b] $\to R$ ,
$I(f) = \int_a^b f$

Proof: $I(sf + tg) = \int_a^b sf + tg = s\int_a^b f + t\int_a^b g = sI(f) + tI(g)$

Example 5.) The inverse of a linear function is linear (when the inverse exists).

Suppose $f^{-1}(x) = c$, $f^{-1}(y) = d$.

Then $f(c) = x$ and $f(d) = y$ and
$f(ac + bd) = af(c) + bf(d) = ax + by$.

Hence $f^{-1}(ax + by) = ac + bd = af^{-1}(x) + bf^{-1}(y)$.

Example 6.) $D$ : set of all twice differential functions $\to$ set of all functions, $L(f) = af'' + bf' + cf$

Proof:

$$\begin{aligned} L(sf + tg) &= a(sf + tg)'' + b(sf + tg)' + c(sf + tg) \\ &= saf'' + tag'' + sbf' + tbg' + scf + tcg \\ &= s(af'' + bf' + cf) + t(ag'' + bg' + cg) \\ &= sL(f) + tL(g) \end{aligned}$$



Consequence 1: If $\psi_1, \psi_2$ are solutions to $af'' + bf' + cf = 0$, then $3\psi_1 + 5\psi_2$ is also a solution to $af'' + bf' + cf = 0$,

Proof: Since $\psi_1, \psi_2$ are solutions to $af''+bf'+cf = 0$, $L(\psi_1) = 0$ and $L(\psi_2) = 0$.

$$\begin{aligned} \text{Hence } L(3\psi_1 + 5\psi_2) &= 3L(\psi_1) + 5L(\psi_2) \\ &= 3(0) + 5(0) = 0. \end{aligned}$$

Thus $3\psi_1 + 5\psi_2$ is also a solution to $af''+bf'+cf = 0$

Consequence 2:
If $\psi_1$ is a solution to $af'' + bf' + cf = h$
and $\psi_2$ is a solution to $af'' + bf' + cf = k$,
then $3\psi_1+5\psi_2$ is a solution to $af''+bf'+cf = 3h+5k$,

Since $\psi_1$ is a solution to $af''+bf'+cf = h$, $L(\psi_1) = h$.

Since $\psi_2$ is a solution to $af''+bf'+cf = k$, $L(\psi_2) = k$.

$$\begin{aligned} \text{Hence } L(3\psi_1 + 5\psi_2) &= 3L(\psi_1) + 5L(\psi_2) \\ &= 3h + 5k. \end{aligned}$$

Thus $3\psi_1 + 5\psi_2$ is also a solution to
$$af'' + bf' + cf = 3h + 5k$$

# Pf #2 of Thm 3.2.2

Note by ex 6;

$L(f) = af'' + bf' + cf$

is a linear function

$ay'' + by' + cy = 0 \quad (\star)$

$\phi_1$ is a soln to $(\star) \Longrightarrow$

$L(\phi_1) = a\phi_1'' + b\phi_1' + c\phi_1 = 0$

$\phi_2$ is a soln to $(\star) \Longrightarrow$

$L(\phi_2) = a\phi_2'' + b\phi_2' + c\phi_2 = 0$

$$\begin{aligned} L(c_1\phi_1 + c_2\phi_2) &= c_1 L(\phi_1) + c_2 L(\phi_2) \\ &= c_1(0) + c_2(0) \\ &= 0 \end{aligned}$$

$\Longrightarrow c_1\phi_1 + c_2\phi_2$ is a soln to $(\star)$

**Second order differential equation:**

Linear equation with constant coefficients:
If the second order differential equation is

$$ay'' + by' + cy = 0,$$

then $y = e^{rt}$ is a solution

Need to have two independent solutions.

Solve the following IVPs:

1.) $y'' - 6y' + 9y = 0$ $\quad y(0) = 1,\ y'(0) = 2$

2.) $4y'' - y' + 2y = 0$ $\quad y(0) = 3,\ y'(0) = 4$

3.) $4y'' + 4y' + y = 0$ $\quad y(0) = 6,\ y'(0) = 7$

4.) $2y'' - 2y = 0$ $\quad y(0) = 5,\ y'(0) = 9$



a, b, c constants

$ay'' + by' + cy = 0, \quad y = e^{rt}$, then
$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $r_1 \neq r_2$, then $b^2 - 4ac \neq 0$. Hence a general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$

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If $b^2 - 4ac > 0$, general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$.

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If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula.

general solution is $y = c_1e^{dt}cos(nt) + c_2e^{dt}sin(nt)$ where $r = d \pm in$

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If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: $te^{r_1t}$

Hence general solution is $y = c_1e^{r_1t} + c_2te^{r_1t}$.

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Initial value problem: use $y(t_0) = y_0$, $y'(t_0) = y'_0$ to solve for $c_1, c_2$ to find unique solution.

Derivation of general solutions:

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If $b^2 - 4ac > 0$ we guessed $e^{rt}$ is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

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Section 3.3: If $b^2 - 4ac < 0$, :

Changed format of $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula:

$$\boxed{e^{it} = cos(t) + isin(t)}$$

Hence $\boxed{e^{(d+in)t} = e^{dt} e^{int} = e^{dt}[cos(nt) + isin(nt)]}$

Let $r_1 = d + in$, $r_2 = d - in$

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$
$$= c_1 e^{dt}[cos(nt) + isin(nt)] + c_2 e^{dt}[cos(-nt) + isin(-nt)]$$
$$= c_1 e^{dt} cos(nt) + ic_1 e^{dt} sin(nt) + c_2 e^{dt} cos(nt) - ic_2 e^{dt} sin(nt)$$
$$= (c_1 + c_2) e^{dt} cos(nt) + i(c_1 - c_2) e^{dt} sin(nt)$$
$$= k_1 e^{dt} cos(nt) + k_2 e^{dt} sin(nt)$$

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Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$.
Hence one solution is $y = e^{r_1 t}$ Need second solution.

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.

How about $y = v(t)e^{rt}$?

$$y' = v'(t)e^{rt} + v(t)re^{rt}$$
$$y'' = v''(t)e^{rt} + v'(t)re^{rt} + v'(t)re^{rt} + v(t)r^2 e^{rt}$$
$$= v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2 e^{rt}$$

$$ay'' + by' + cy = 0$$

$$a(v''e^{rt} + 2v're^{rt} + vr^2 e^{rt}) + b(v'e^{rt} + vre^{rt}) + cve^{rt} = 0$$

$$a(v''(t) + 2v'(t)r + v(t)r^2) + b(v'(t) + v(t)r) + cv(t) = 0$$

$$av''(t) + 2av'(t)r + av(t)r^2 + bv'(t) + bv(t)r + cv(t) = 0$$

$$av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0$$

$$av''(t) + (2a(\tfrac{-b}{2a}) + b)v'(t) + 0 = 0$$

since $ar^2 + br + c = 0$ and $r = \frac{-b}{2a}$

$av''(t) + (-b + b)v'(t) = 0.$ Thus $av''(t) = 0$.

Hence $v''(t) = 0$ and $v'(t) = k_1$ and $v(t) = k_1 t + k_2$

Hence $v(t)e^{r_1 t} = (k_1 t + k_2)e^{r_1 t}$ is a soln

Thus $te^{r_1 t}$ is a nice second solution.

Hence general solution is $y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$

Solve: $y'' + y = 0$, $y(0) = -1$, $y'(0) = -3$

$r^2 + 1 = 0$ implies $r^2 = -1$. Thus $r = \pm i$.

RECOMMENDED Method:

Since $r = 0 \pm 1i$ $\boxed{y = k_1 cos(t) + k_2 sin(t)}$

Then $y' = -k_1 sin(t) + k_2 cos(t)$

$y(0) = -1$: $-1 = k_1 cos(0) + k_2 sin(0)$ implies $-1 = k_1$

$y'(0) = -3$: $-3 = -k_1 sin(0) + k_2 cos(0)$ implies $-3 = k_2$

Thus IVP solution: $\boxed{y = -cos(t) - 3sin(t)}$

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NOT RECOMMENDED: work with $y = c_1 e^{it} + c_2 e^{-it}$

$r = \pm i$

$y' = ic_1 e^{it} - ic_2 e^{-it}$

$y(0) = -1$: $-1 = c_1 e^0 + c_2 e^0$ implies $-1 = c_1 + c_2$.

$y'(0) = -3$: $-3 = ic_1 e^0 - ic_2 e^0$ implies $-3 = ic_1 - ic_2$.

$\times i$

$-1i = ic_1 + ic_2$.

$-3 = ic_1 - ic_2$.

$2ic_1 = -3 - i$ implies $c_1 = \frac{-3i - i^2}{-2} = \frac{3i-1}{2}$

$i\frac{(2ic_2)}{-2} = \frac{(3-i)i}{-2}$ implies $c_2 = \frac{3i - i^2}{-2} = \frac{-3i-1}{2}$

Euler's formula: $\boxed{e^{ix} = cos(x) + isin(x)}$

$y = (\frac{3i-1}{2})e^{it} + (\frac{-3i-1}{2})e^{-it} = (\frac{3i-1}{2})[cos(t) + isin(t)] + (\frac{-3i-1}{2})[cos(-t) + isin(-t)]$

$= (\frac{3i-1}{2})[cos(t) + isin(t)] + (\frac{-3i-1}{2})[cos(t) - isin(t)]$

$= \cancel{(\frac{3i}{2})cos}(t) + (\frac{3i}{2})isin(t) + (\frac{-1}{2})cos(t) + \cancel{(\frac{-1}{2})isin}(t) + \cancel{(\frac{-3i}{2})cos}(t) - (\frac{-3i}{2})isin(t) + (\frac{-1}{2})cos(t) - \cancel{(\frac{-1}{2})isin}(t)$

$= (\frac{3i}{2})isin(t) + (\frac{-1}{2})cos(t) + (\frac{3i}{2})isin(t) + (\frac{-1}{2})cos(t)$

$= -(\frac{3}{2})sin(t) - (\frac{1}{2})cos(t) - (\frac{3}{2})sin(t) - (\frac{1}{2})cos(t)$

$\boxed{y = -3sin(t) - 1cos(t)}$

2) $4y'' - y' + 2y = 0$

$y = e^{rt}$

$4r^2 - r + 2 = 0$

$$r = \frac{+1 \pm \sqrt{1 - 4(4)(2)}}{2(4)} = \frac{1 \pm \sqrt{-31}}{8}$$

$$r_1, r_2 = \frac{1 \pm i\sqrt{31}}{8} = \frac{1}{8} \pm i\left(\frac{\sqrt{31}}{8}\right)$$

$d \pm i n$

$d = \frac{1}{8} \quad n = \frac{\sqrt{31}}{8}$

$$\boxed{\text{Gen soln: } y = c_1 e^{t/8} \cos\left(\frac{\sqrt{31}}{8} t\right) + c_2 e^{t/8} \sin\left(\frac{\sqrt{31}}{8} t\right)}$$

$y(0) = 3, \quad y'(0) = 4$

$y' = \quad \Rightarrow$

$$y = e^{t/8}\left(c_1 \cos\left(\frac{\sqrt{31}}{8}t\right) + c_2 \sin\left(\frac{\sqrt{31}}{8}t\right)\right)$$

$$y' = \frac{1}{8}e^{t/8}\left(c_1 \cos\left(\frac{\sqrt{31}}{8}t\right) + c_2 \sin\left(\frac{\sqrt{31}}{8}t\right)\right)$$

$$+ e^{t/8}\left[-c_1\left(\frac{\sqrt{31}}{8}\right) \sin\left(\frac{\sqrt{31}}{8}t\right) + c_2\left(\frac{\sqrt{31}}{8}\right)\cos\left(\frac{\sqrt{31}}{8}t\right)\right]$$

$y(0)=3:\quad 3 = c_1 + c_2(0)$

$y'(0)=4:\quad 4 = \frac{1}{8}c_1 + c_2\left(\frac{\sqrt{31}}{8}\right)$

$$\Rightarrow \quad c_1 = 3$$

$$\frac{\sqrt{31}}{8}c_2 = 4 - \frac{1}{8}(3)$$

$$\sqrt{31}\, c_2 = 32 - 3 = 29$$

$$c_2 = 29/\sqrt{31}$$

IVP soln

$$\boxed{y = 3e^{t/8}\cos\left(\frac{\sqrt{31}}{8}t\right) + \frac{29}{\sqrt{31}}e^{t/8}\sin\left(\frac{\sqrt{31}}{8}t\right)}$$

$$4y'' + 4y' + y = 0$$

$$4r^2 + 4r + 1 = 0$$

$$(2r+1)(2r+1) = 0 \Rightarrow r = -\frac{1}{2}$$

$$\boxed{y = c_1 e^{-\frac{1}{2}t} + c_2 t e^{-\frac{1}{2}t}}$$

$$y(0) = 6, \quad y'(0) = 7$$

$$y' = -\frac{1}{2} c_1 e^{-t/2} + c_2 \left[ e^{-t/2} + t\left(-\frac{1}{2} e^{-t/2}\right) \right]$$

$y(0) = 6$: $\quad 6 = c_1 + 0$

$y'(0) = 7$: $\quad 7 = -\frac{1}{2} c_1 + c_2$

$$\Rightarrow c_1 = 6, \quad c_2 = 7 + \frac{1}{2}(6) = 10$$

IVP soln: $\boxed{y = 6e^{-t/2} + 10te^{-t/2}}$

## 3.3: Linear Independence and the Wronskian

Defn: $f$ and $g$ are linearly dependent if there exists constants $c_1, c_2$ such that $c_1 \neq 0$ or $c_2 \neq 0$ and $c_1 f(t) + c_2 g(t) = 0$ for all $t \in (a, b)$

Thm 3.3.1: If $f : (a, b) \to R$ and $g(a, b) \to R$ are differentiable functions on (a, b) and if $W(f, g)(t_0) \neq 0$ for some $t_0 \in (a, b)$, then $f$ and $g$ are linearly independent on $(a, b)$. Moreover, if $f$ and $g$ are linearly dependent on $(a, b)$, then $W(f, g)(t) = 0$ for all $t \in (a, b)$

If $c_1 f(t) + c_2 g(t) = 0$ for all $t$, then $c_1 f'(t) + c_2 g'(t) = 0$

Solve the following linear system of equations for $c_1, c_2$

$$\begin{aligned} c_1 f(t_0) + c_2 g(t_0) &= 0 \\ c_1 f'(t_0) + c_2 g'(t_0) &= 0 \end{aligned}$$

$$\begin{bmatrix} f(t_0) & g(t_0) \\ f'(t_0) & g'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$