3.1 - 3.4: Homogeneous linear differential equation

$$a(t)y'' + b(t)y' + c(t)y = 0$$

linear combination of y, y', and y''

homogeneous

Thm 3.2.2: If $y = \phi_1(t)$ and $y = \phi_2(t)$ are two solutions to a homogeneous linear differential equation, then

$$y = c_1\phi_1(t) + c_2\phi_2(t)$$

is also a solution to this homogeneous linear differential equation.

Proof of thm 3.2.2: Plug $y = c_1\phi_1(t) + c_2\phi_2(t)$ to see that it satisfies the homogeneous linear differential equation.

To solve homogeneous linear differential equation with constant coefficients

$$ay'' + by' + cy = 0$$

Guess $y = e^{rt}$ for HOMOGENEOUS equation and plug in:

$y' = re^{rt}$, $y'' = r^2e^{rt}$

$ay'' + by' + cy = 0$

$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0 \quad \Rightarrow \quad e^{rt}(ar^2 + br + c) = 0$

$e^{rt} \neq 0$, thus can divide both sides by e^{rt}: $\quad ar^2 + br + c = 0$

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $r_1 \neq r_2$, then $b^2 - 4ac \neq 0$. Hence a general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$

If $b^2 - 4ac > 0$, general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$.

If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula.

general solution is $y = c_1e^{xt}cos(yt) + c_2e^{xt}sin(yt)$ where $r = x \pm iy$

If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: te^{r_1t}

Hence general solution is $y = c_1e^{r_1t} + c_2te^{r_1t}$.

Example 1: $4y'' + y' = 0 \quad \Rightarrow \quad 4r^2 + r = 0$

$4r^2 + r = r(4r+1) = 0$. Thus $r = 0, -\frac{1}{4}$

Hence $y = e^{0t} = 1$ and $y = e^{-\frac{1}{4}t}$ are both solutions.

Thus the general solution is $y = c_1(1) + c_2(e^{-\frac{t}{4}})$

Example 2: $4y'' + y = 0 \quad \Rightarrow \quad 4r^2 + 1 = 0$

$4r^2 + 1 = 0 \quad \Rightarrow \quad 4r^2 = -1$. Thus $r = \pm\frac{i}{2} = 0 \pm \frac{1}{2}i$

Thus the general solution is $y = c_1(cos(\frac{t}{2})) + c_2(sin(\frac{t}{2}))$

Example 3: $4y'' + 2y' + y = 0 \quad \Rightarrow \quad 4r^2 + 2r + 1 = 0$

Thus $r = \frac{-2\pm\sqrt{2^2-4(4)(1)}}{2(4)} = \frac{-2\pm\sqrt{2^2[1-(4)(1)]}}{2(4)}$

$= \frac{-2\pm2\sqrt{-3}}{2(4)} = \frac{2(-1\pm i\sqrt{3})}{2(4)} = \frac{-1\pm i\sqrt{3}}{4} = \frac{-1}{4} \pm i\frac{\sqrt{3}}{4}$

Thus the general solution is $y = c_1(e^{\frac{-t}{4}}cos(\frac{\sqrt{3}}{4}t)) + c_2(e^{\frac{-t}{4}}cos(\frac{\sqrt{3}}{4}t))$

Example 4: $4y'' + 4y' + y = 0 \quad \Rightarrow \quad 4r^2 + 4r + 1 = 0$

$4r^2 + 4r + 1 = (2r+1)(2r+1) = 0$. Thus $r = -\frac{1}{2}, -\frac{1}{2}$

Hence $y = e^{-\frac{1}{2}t}$ is a solution.

NEED 2 linearly independent solutions for a 2nd order linear homogeneous DE.

To get 2nd solution in repeated root case: multiply by t

Thus 2 solutions are $y = e^{-\frac{1}{2}t}$ and $y = te^{-\frac{1}{2}t}$

Thus the general solution is $y = c_1(e^{-\frac{1}{2}t}) + c_2(te^{-\frac{1}{2}t})$

Do your 3.1, 3.3, 3.4 homework NOW!!
On exam 1, exam 2, and final exam.

LEARN IT NOW!!!