Trig background:

$$cos(y \mp x) = cos(x \mp y) = cos(x)cos(y) \pm sin(x)sin(y)$$

Let $A = Rcos(\delta)$, $B = Rsin(\delta)$ in

$$\begin{aligned} Acos(\omega_0 t) + Bsin(\omega_0 t) \\ &= Rcos(\delta)cos(\omega_0 t) + Rsin(\delta)sin(\omega_0 t) \\ &= Rcos(\omega_0 t - \delta) \end{aligned}$$

# 3.7/8 Mechanical Vibrations:

$$mu''(t) + \gamma u'(t) + ku(t) = F_{external}, \quad m, \gamma, k \geq 0$$

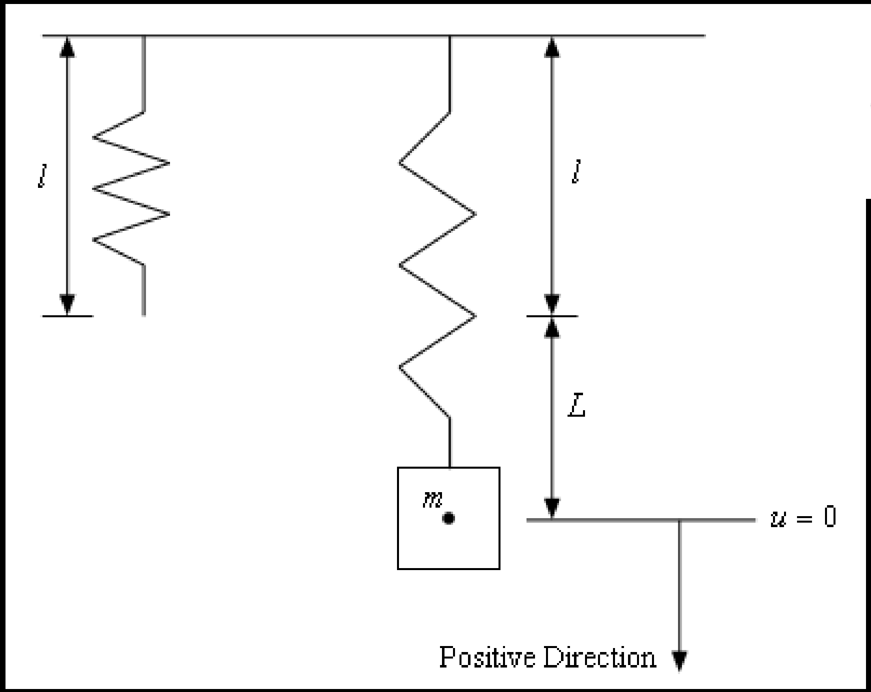
$$mg - kL = 0, \qquad F_{damping}(t) = -\gamma u'(t)$$

$m$= mass,

$k$ = spring force proportionality constant,

$\gamma$ = damping force proportionality constant

$g$ = 9.8 m/sec$^2$ or 32 ft/sec$^2$. Weight = $mg$.




$$mu'' = mg - k(L + u) - \gamma u' + F(t)$$

https://tutorial.math.lamar.edu/Classes/DE/Vibrations.aspx

Electrical Vibrations:

Voltage drop across inductor + resistor + capacitor
= the supplied voltage

$$L\frac{dI(t)}{dt} + RI(t) + \frac{1}{C}Q(t) = E(t), \quad L, R, C \geq 0 \text{ and } I = \frac{dQ}{dt}$$

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t)$$

$L$ = inductance (henrys),

$R$ = resistance (ohms)

$C$ = capacitance (farads)

$Q(t)$ = charge at time $t$ (coulombs)

$I(t)$ = current at time $t$ (amperes)

$E(t)$ = impressed voltage (volts).

1 volt = 1 ohm · 1 ampere = 1 coulomb / 1 farad =
1 henry · 1 amperes/ 1 second

Suppose a mass weighing 64 lbs stretches a spring 4 ft. If there is no damping and the spring is stretched an additional foot and set in motion with an upward velocity of $\sqrt{8}$ ft/sec, find the equation of motion of the mass.

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$$Weight = mg\text{: } m = \frac{weight}{g} = \frac{64}{32} = 2$$

$$mg - kL = 0 \text{ implies } k = \frac{mg}{L} = \frac{64}{4} = 16$$

$$mu''(t) + \gamma u'(t) + ku(t) = F_{external}$$

$$2u''(t) + 16u(t) = 0$$

$$u''(t) + 8u(t) = 0, \qquad u(0) = 1,\ u'(0) = -\sqrt{8}$$

$$r^2 + 8 = 0 \to r = \pm\sqrt{-8} = \pm i\sqrt{8} = 0 \pm i\sqrt{8}$$

$$u(t) = c_1 e^{it\sqrt{8}} + c_2 e^{-it\sqrt{8}}$$

$$u''(t) + 8u(t) = 0$$

$$u(0) = 1,\ u'(0) = -\sqrt{8}$$

$$r^2 + 8 = 0 \rightarrow r^2 = -8 \rightarrow r = \sqrt{-8} = i\sqrt{8} = 0 + i\sqrt{8}$$

$$u(t) = e^{-\frac{\gamma t}{2m}}(Acos\mu t + Bsin\mu t)$$

$$u(t) = Acos\sqrt{8}t + Bsin\sqrt{8}t$$

$$u(0) = 1\text{: } 1 = Acos(0) + Bsin(0) = A$$

$$u'(t) = -\sqrt{8}Asin\sqrt{8}t + \sqrt{8}Bcos\sqrt{8}t$$

$$u'(0) = -\sqrt{8} : -\sqrt{8} = -\sqrt{8}Asin(0) + \sqrt{8}Bcos(0)$$

$$B = -1$$

$$u(t) = cos\sqrt{8}t - sin\sqrt{8}t$$

$$u(t) = cos(t\sqrt{8}) - sin(t\sqrt{8}) = Rcos(t\sqrt{8} - \delta)$$

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Amplitude $= R$

frequency $= \omega_0$ (measured in radians per unit time).

period $= \frac{2\pi}{\omega_0}$ phase (displacement) $= \delta$

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frequency $= \omega_0$ (measured in radians per unit time).

period $= \frac{2\pi}{\omega_0}$ phase (displacement) $= \delta$

$A = Rcos(\delta)$, $B = Rsin(\delta)$ implies

$$\begin{aligned} A^2 + B^2 &= R^2cos^2(\delta) + R^2sin^2(\delta) \\ &= R^2(cos^2(\delta) + sin^2(\delta)) = R^2 \end{aligned}$$

$c_1 = Rcos(\delta)$, $c_2 = Rsin(\delta)$ implies

$$c_1^2 + c_2^2 = R^2 cos^2(\delta) + R^2 sin^2(\delta)$$
$$= R^2(cos^2(\delta) + sin^2(\delta)) = R^2$$

and $\frac{Rsin(\delta)}{Rcos(\delta)} = tan(\delta) = \frac{c_2}{c_1}$

BUT easier to plot to convert Euclidean coordinates $(c_1, c_2) = (Rcos(\delta), Rsin(\delta))$ into polar coordinates $(R, \delta) =$ (length, angle).

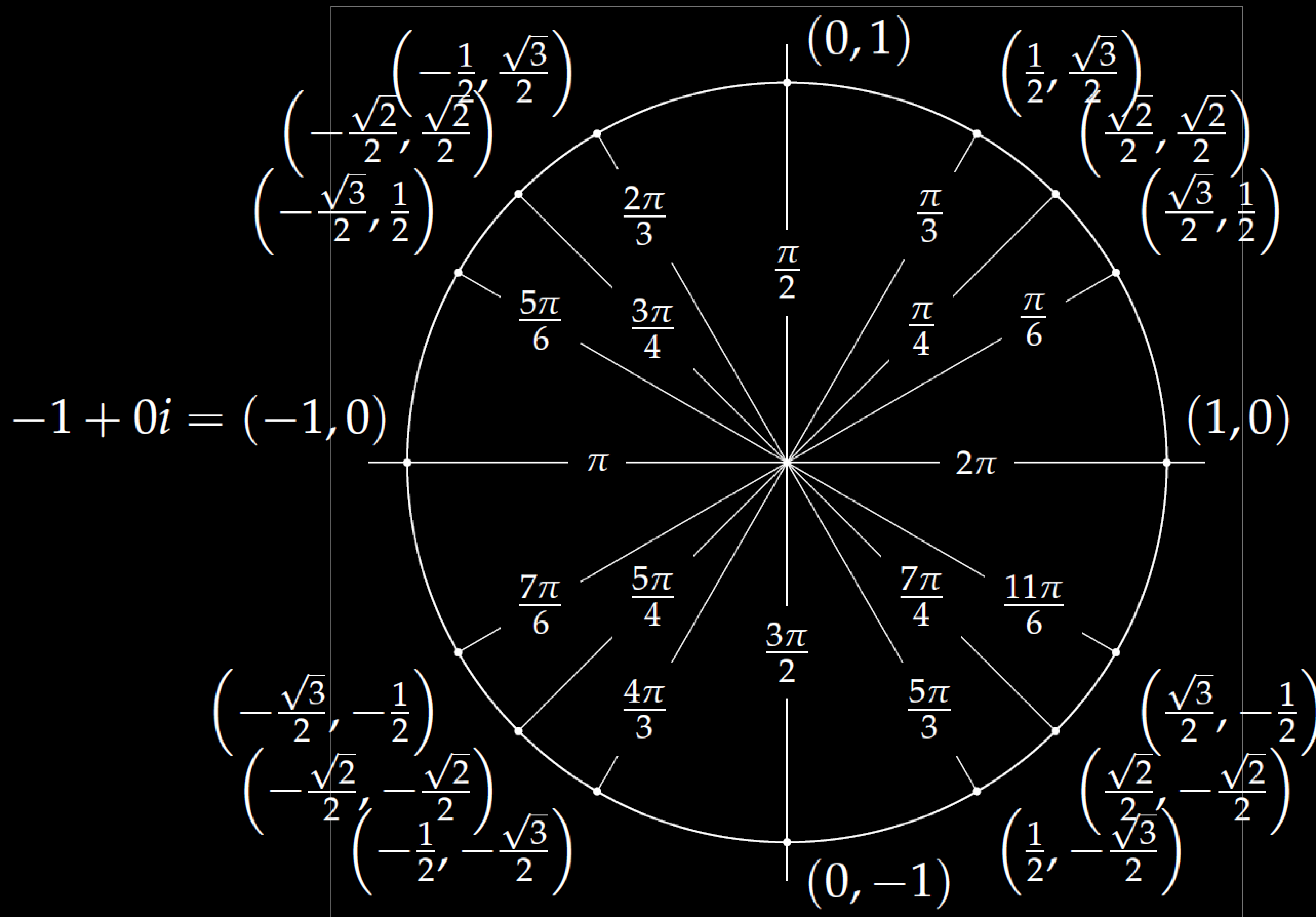
$$sin(0) = \frac{\sqrt{0}}{2} = 0 \qquad cos(0) = 1$$

$$sin(\tfrac{\pi}{6}) = \frac{\sqrt{1}}{2} = \frac{1}{2} \qquad cos(\tfrac{\pi}{6}) = \frac{\sqrt{3}}{2}$$

$$sin(\tfrac{\pi}{4}) = \frac{\sqrt{2}}{2} \qquad cos(\tfrac{\pi}{4}) = \frac{\sqrt{2}}{2}$$

$$sin(\tfrac{\pi}{3}) = \frac{\sqrt{3}}{2} \qquad cos(\tfrac{\pi}{3}) = \frac{1}{2}$$

$$sin(\tfrac{\pi}{2}) = \frac{\sqrt{4}}{2} = 1 \qquad cos(\tfrac{\pi}{2}) = 0$$



$-1+0i=(-1,0)$
$(0,1)$
$(1,0)$
$(0,-1)$
$\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$
$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$
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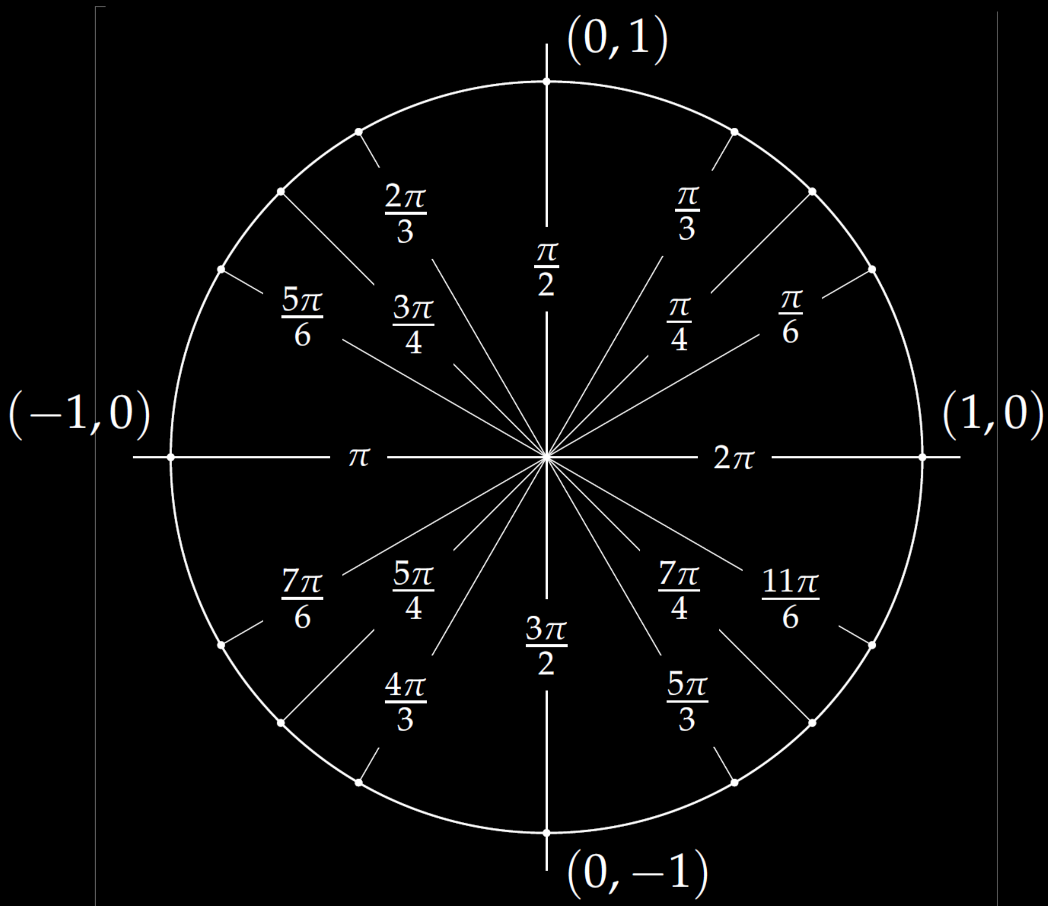
$\pi$
$2\pi$
$\frac{7\pi}{6}$
$\frac{5\pi}{4}$
$\frac{4\pi}{3}$
$\frac{3\pi}{2}$
$\frac{5\pi}{3}$
$\frac{7\pi}{4}$
$\frac{11\pi}{6}$

$$u(t) = cos(t\sqrt{8}) - sin(t\sqrt{8}) = Rcos(t\sqrt{8} - \delta)$$

$$= \sqrt{2}cos(t\sqrt{8} - \frac{7\pi}{4}) = \sqrt{2}cos(t\sqrt{8} + \frac{\pi}{4})$$

$$A = 1 = Rcos(\delta) \text{ and } B = -1 = Rsin(\delta)$$



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Thus $R = \sqrt{A^2 + B^2} = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$

$tan(\delta) = \frac{Rsin(\delta)}{Rcos(\delta)} = \frac{B}{A} = \frac{-1}{1}$. Thus $\delta = -\frac{\pi}{4} = \frac{7\pi}{4}$

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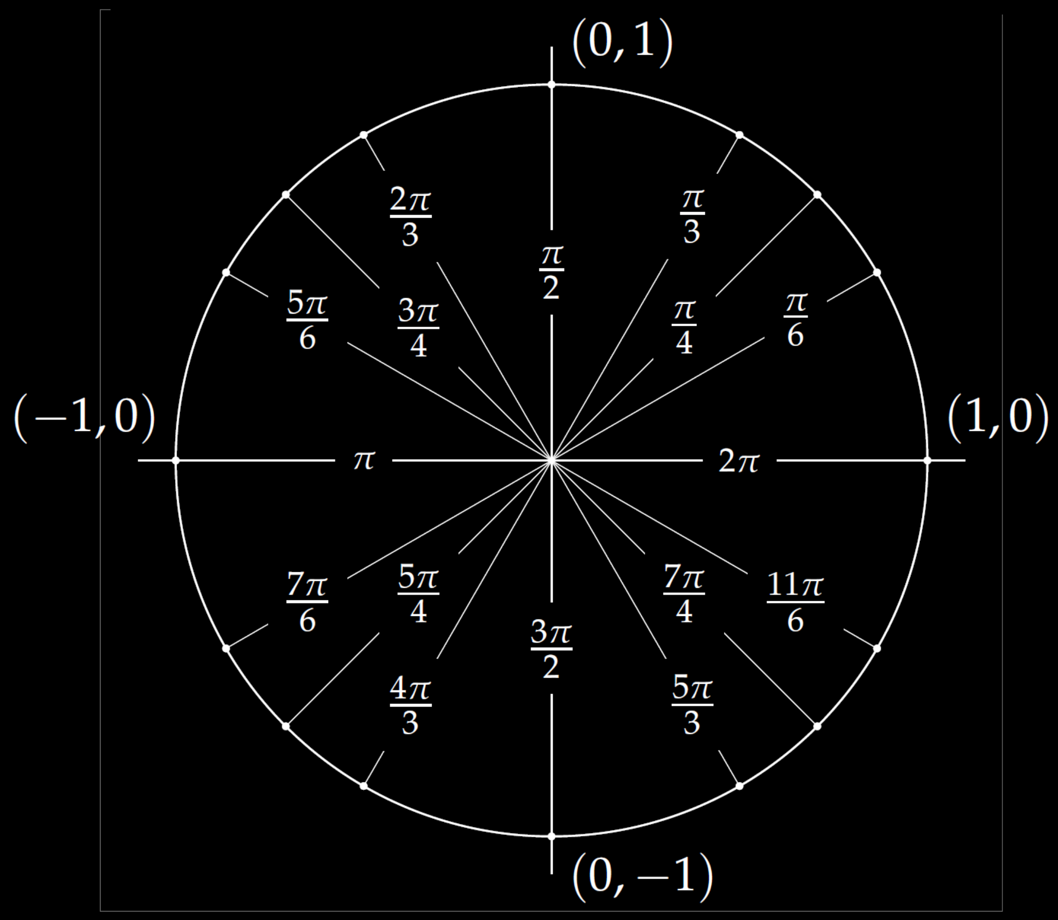

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