3.5: Non-homogeneous linear differential equation

$$a(t)y'' + b(t)y' + c(t)y = g(t) \qquad \text{where } g(t) \neq 0$$

Step 1: Find homogeneous general solution to $a(t)y'' + b(t)y' + c(t)y = 0$:

$$y = c_1\phi_1(t) + c_2\phi_2(t)$$

Step 2: Find one non-homogeneous solution to $a(t)y''+b(t)y'+c(t)y = g(t)$:

Step 3: Combine these solutions to create general solution to the non-homogeneous linear DE:

$$y = c_1\phi_1(t) + c_2\phi_2(t) + \psi$$

Compare to linear algebra: non-homogenous solution is a "shifted" version of the homogeneous solution (suppose solution space is 2-dimensional):

$$\begin{array}{rcl} A\mathbf{x} = \mathbf{0} & \Rightarrow & \mathbf{x} = c_1\mathbf{v_1} + c_2\mathbf{v_2} \\ A\mathbf{x} = \mathbf{b} & \Rightarrow & \mathbf{x} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \mathbf{w} \end{array}$$

$$\begin{array}{rcl} a(t)y'' + b(t)y' + c(t)y = 0 & \Rightarrow & y = c_1\phi_1(t) + c_2\phi_2(t) \\ a(t)y'' + b(t)y' + c(t)y = g(t) & \Rightarrow & y = c_1\phi_1(t) + c_2\phi_2(t) + \psi \end{array}$$

Example 1: Solve $y'' - 4y' + 4y = e^{3t}$

Step 1: Solve homogeneous version:

$y'' - 4y' + 4y = 0 \quad \Rightarrow \quad r^2 - 4r + 4 = 0 \quad \Rightarrow \quad (r-2)^2 = 0 \quad \Rightarrow \quad r = 2, 2$

Thus general solution to homogeneous DE is $y = c_1e^{2t} + c_2te^{2t}$

Step 2: Find ONE non-homogeneous solution to $y'' - 4y' + 4y = e^{3t}$

3.5 Method of undetermined coefficients – Educated guess: $y = Ae^{3t}$

Plug in to solve for undetermined coefficient A:

$$y = Ae^{3t} \quad \Rightarrow \quad y' = 3Ae^{3t} \quad \Rightarrow \quad y'' = 9Ae^{3t}$$

$9Ae^{3t} - 4(3Ae^{3t}) + 4Ae^{3t} = e^{3t} \quad \Rightarrow \quad 9A - 12A + 4A = 1 \quad \Rightarrow \quad A = 1$

Thus ONE non-homogeneous solution is $y = e^{3t}$

Step 3: Combine these solutions to create general solution to the non-homogeneous linear DE:

The general solution to nonhomogeneous DE is $y = c_1e^{2t} + c_2te^{2t} + e^{3t}$

Example 2: Solve $y'' - 4y' + 4y = 2cos(3t)$

Step 1: Solve homogeneous version:

$y'' - 4y' + 4y = 0 \quad \Rightarrow \quad r^2 - 4r + 4 = 0 \quad \Rightarrow \quad (r-2)^2 = 0 \quad \Rightarrow \quad r = 2, 2$

Thus general solution to homogeneous DE is $y = c_1 e^{2t} + c_2 t e^{2t}$

Step 2: Find ONE non-homogeneous solution to $y'' - 4y' + 4y = 2cos(3t)$

3.5 Method of undetermined coefficients – Educated guess: $y = Acos(3t) + Bsin(3t)$

Plug in to solve for undetermined coefficient A: $y = Acos(3t) + Bsin(3t)$
$\Rightarrow \quad y' = -3Asin(3t) + 3Bcos(3t) \quad \Rightarrow \quad y'' = -9Acos(3t) - 9Bsin(3t)$

$$\begin{aligned} y'' &= -9Acos(3t) - 9Bsin(3t) \\ -4y' &= -12Bcos(3t) + 12Asin(3t) \quad \text{– Note switched order of } B \text{ and } A \text{ term} \\ +4y &= +4Acos(3t) + 4Bsin(3t) \end{aligned}$$

$y'' - 4y' + 4y = (-9A - 12B + 4A)cos(3t) + (-9B + 12A + 4B)sin(3t)$

$y'' - 4y' + 4y = (-5A - 12B)cos(3t) + (12A - 5B)sin(3t)$

$y'' - 4y' + 4y = 2cos(3t)$

$(-5A - 12B)cos(3t) + (12A - 5B)sin(3t) = 2cos(3t)$

$-5A - 12B = 2$ and $12A - 5B = 0$

From 2nd equation: $5B = 12A$ and $B = \frac{12A}{5}$

Plug into first equation: $-5A - 12(\frac{12A}{5}) = 2$

$-25A - 144A = 10 \quad \Rightarrow \quad -169A = 10 \quad \Rightarrow \quad A = -\frac{10}{169}$

Hence $B = (\frac{12}{5})A = B = (\frac{12}{5})(-\frac{10}{169}) = -(\frac{12}{1})(\frac{2}{169}) = -\frac{24}{169}$

Thus ONE non-homogeneous solution is $y = -\frac{10}{169}cos(3t) - \frac{24}{169}sin(3t)$

Step 3: Combine these solutions to create general solution to the non-homogeneous linear DE:

The general solution to nonhomogeneous DE is

$$y = c_1 e^{2t} + c_2 t e^{2t} + -\frac{10}{169}cos(3t) - \frac{24}{169}sin(3t)$$

Give the form of the particular (nonhomogenous) solution with undetermined coefficients for

$$y'' + 4y' + 4y = sin(3t)$$

$Y(t) =$ ______________________________ (Do NOT solve!)

Give the form of the particular (nonhomogenous) solution with undetermined coefficients for

$$y'' + 4y' + 4y = 4sin(3t)$$

$Y(t) =$ ______________________________ (Do NOT solve!)

Give the form of the particular (nonhomogenous) solution with undetermined coefficients for

$$y'' + 4y' + 4y = sin(3t) - 8cos(3t)$$

$Y(t) =$ ______________________________ (Do NOT solve!)

Answers:

Give the form of the particular (nonhomogenous) solution with undetermined coefficients for

$$y'' + 4y' + 4y = sin(3t)$$

$Y(t) =$ $\underline{\ y = Acos(3t) + Bsin(3t)\ }$ (Do NOT solve!)

Give the form of the particular (nonhomogenous) solution with undetermined coefficients for

$$y'' + 4y' + 4y = 4sin(3t)$$

$Y(t) =$ $\underline{\ y = Acos(3t) + Bsin(3t)\ }$ (Do NOT solve!)

Give the form of the particular (nonhomogenous) solution with undetermined coefficients for

$$y'' + 4y' + 4y = sin(3t) - 8cos(3t)$$

$Y(t) =$ $\underline{\ y = Acos(3t) + Bsin(3t)\ }$ (Do NOT solve!)