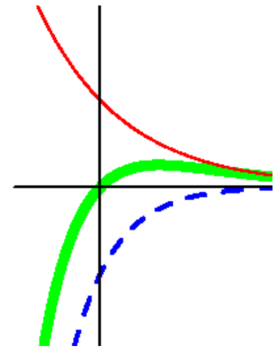
1. The following shows the graphs of $u = e^{-t}$, $u = -e^{-2t}$ and $u = e^{-t} - e^{-2t}$.



1a.) The thin red curve corresponds to $\underline{\ u = e^{-t}\ }$.

1b.) The medium dotted blue curve corresponds to $\underline{\ u = -e^{-2t}\ }$.

1c.) The thick light green curve corresponds to $\underline{\ u = e^{-t} - e^{-2t}\ }$.

1d.) $\lim_{t \to -\infty} e^{-t} - e^{-2t} = -\infty$

1e.) $\lim_{t \to \infty} e^{-t} - e^{-2t} = 0$

1f.) As $t \to -\infty$ which term of $e^{-t} - e^{-2t}$ dominates. In other words, does $e^{-t} - e^{-2t}$ look more like e^{-t} or $-e^{-2t}$ for large negative values of t

$$-e^{-2t}$$

1f.) As $t \to \infty$ which term of $e^{-t} - e^{-2t}$ dominates. In other words, does $e^{-t} - e^{-2t}$ look more like e^{-t} or $-e^{-2t}$ for large positive values of t

$$e^{-t}$$

2a.) Suppose $u(t) = c_1\phi_i + c_1\phi_2$ is the general **homogeneous** solution to a differential equation modeling mechanical vibration **with damping**, then $\lim_{t \to \infty} u(t) = 0$

2b.) Will the initial values affect the long-term behaviour of the general **homogeneous** solution to a differential equation modeling mechanical vibration **with damping**?

No.

3.) Suppose $u''+4u = tcos(2t)$ models mechanical vibration (with no damping and with external force $= tcos(2t)$)

Normally when solving $au'' + bu' + cu = tcos(2t)$ one would guess a non-homogenous solution to be of the form $u = (At + B)[Ccos(2t) + Dsin(2t)]$ since $tcos(2t)$ is the product of a degree 1 polynomial and $cos(2t)$. But that won't work in this case.

$(At + B)[Ccos(2t) + Dsin(2t)] = (At)[Ccos(2t) + Dsin(2t)] + (B)[Ccos(2t) + Dsin(2t)]$

3a.) $u = (B)[Ccos(2t)+Dsin(2t)]$ is NOT a solution to the non-homogeneous equation $u''+4u = tcos(2t)$ since it is a solution to the equation:

$$u'' + 4u = 0$$

Sidenote 1: If you did include this in your guess, this term would cancel out since it is a homogeneous solution, and you would see that any choice of B would work IF the rest of your guess worked. $B = 0$ is the simplest choice.

3b.) $u = (At)[Ccos(2t)+Dsin(2t)]$ is NOT a solution to the non-homogeneous equation $u''+4u = tcos(2t)$ since it is a solution to the equation:

$$u'' + 4u = cos(2t)$$

Sidenote 2: If I were solving the equation that answers 3B, I would let C and D swallow the constant A and thus guess $u = t[Ccos(2t) + Dsin(2t)]$ instead of $u = (At)[Ccos(2t) + Dsin(2t)]$

3c.) Thus the best guess for a non-homogeneous solution to $u'' + 4u = tcos(2t)$ would be

$u = t(At + B)[Ccos(2t) + Dsin(2t)]$ or $u = A_1t^2cos(2t) + A_2t^2sin(2t) + A_3tcos(2t) + A_4tsin(2t)$

Hint: Your answer will likely include 4 unknowns.

3d.) Plugging in $u = t(At + B)[Ccos(2t) + Dsin(2t)]$ would result in 4 nonlinear equations involving products such as AC. Note

$$\begin{aligned} u &= t(At + B)[Ccos(2t) + Dsin(2t)] \\ &= ACt^2cos(2t) + ADt^2sin(2t) + BCtcos(2t) + BDtsin(2t) \\ &= A_1t^2cos(2t) + A_2t^2sin(2t) + A_3tcos(2t) + A_4tsin(2t) \end{aligned}$$

Thus we can instead plug in $u = A_1t^2cos(2t) + A_2t^2sin(2t) + A_3tcos(2t) + A_4tsin(2t)$.

Solve $u'' + 4u = tcos(2t)$ using that

$$[A_1t^2cos(2t) + A_2t^2sin(2t) + A_3tcos(2t) + A_4tsin(2t)]'' \\ +4[A_1t^2cos(2t) + A_2t^2sin(2t) + A_3tcos(2t) + A_4tsin(2t)]$$

$$= -8A_1tsin(2t) + 2A_2sin(2t) - 4A_3sin(2t) + 8A_2tcos(2t) + 2A_1cos(2t) + 4A_4cos(2t)$$

Answer: The LHS of $u'' + 4u = tcos(2t)$ is given above. Setting LHS = RHS:

$$-8A_1tsin(2t) + 2A_2sin(2t) - 4A_3sin(2t) + 8A_2tcos(2t) + 2A_1cos(2t) + 4A_4cos(2t) = tcos(2t)$$

Combine like terms:

$$-8A_1tsin(2t) + (2A_2 - 4A_3)sin(2t) + 8A_2tcos(2t) + (2A_1 + 4A_4)cos(2t)$$
$$= 0tsin(2t) + 0sin(2t)tcos(2t) + 0cos(2t)$$

Since $\{tsin(2t), sin(2t), tcos(2t), cos(2t)\}$ is a linearly independent set, the coefficients of like terms from LHS equal the coefficients of the corresponding like terms on RHS. Thus

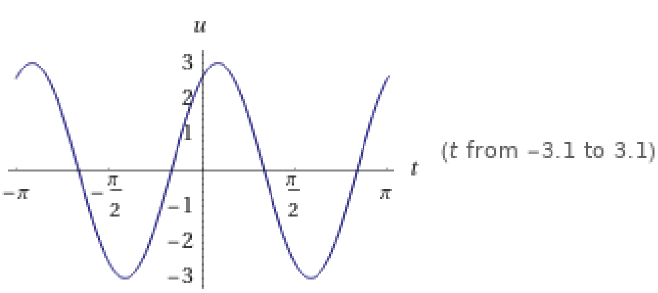
$8A_1 = 0, \quad 2A_2 - 4A_3 = 0, \quad 8A_2t = 1, \quad 2A_1 + 4A_4 = 0$

Thus $A_1 = A_4 = 0$, $A_2 = \frac{1}{8}$, and $A_3 = \frac{1}{2}A_2 = (\frac{1}{2})(\frac{1}{8}) = \frac{1}{16}$

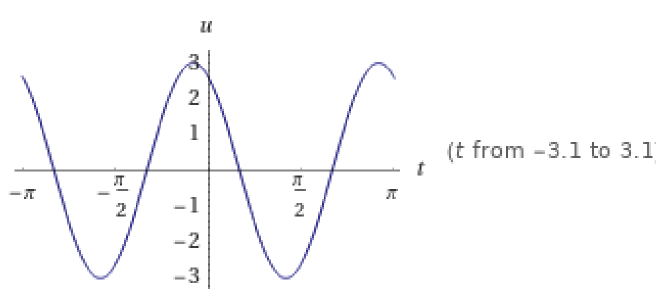
Thus the general solution is $u(t) = c_2sin(2t) + c_1cos(2t) + \frac{1}{8}t^2sin(2t) + \frac{1}{16}tcos(2t)$

4.) The following are solutions to a second order differential equation modeling mechanical vibration. Match these equations to their graphs. State whether the graph corresponds to an undamped, underdamped, critically damped, or overdamped mechanical vibration.

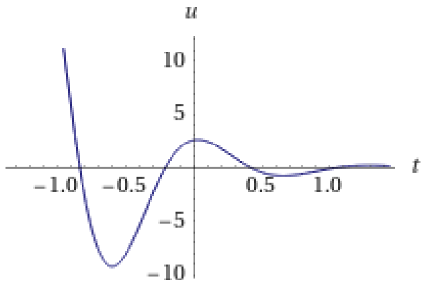
Equation	Graph	damping
$u = 3cos(2t + \frac{\pi}{6})$	B	none
$u = 3cos(2t - \frac{\pi}{6})$	A	none
$u = 3e^{-2t}cos(2t + \frac{\pi}{6})$	F	underdamped
$u = 3e^{-2t}cos(2t - \frac{\pi}{6})$	D	underdamped
$u = 3e^{-2t}cos(5t + \frac{\pi}{6})$	E	underdamped
$u = 3e^{-2t}cos(5t - \frac{\pi}{6})$	C	underdamped
$u = e^{-2t}(1 + t)$	G	critically damped
$u = 100e^{-2t} + e^{-5t}$	I	overdamped
$u = -100e^{-2t} - e^{-5t}$	K	overdamped
$u = -100e^{-2t} + e^{-5t}$	H	overdamped
$u = 100e^{-2t} - e^{-5t}$	J	overdamped



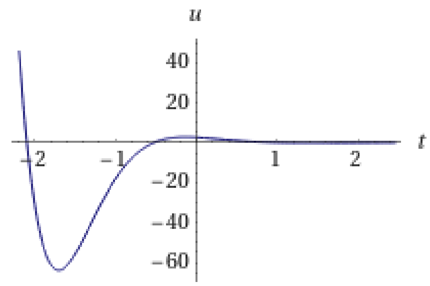
A.



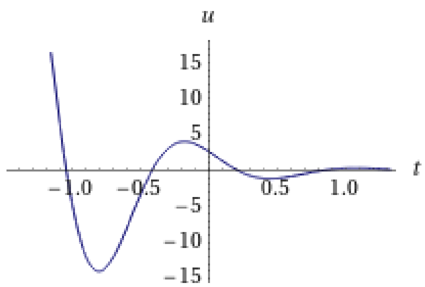
B.



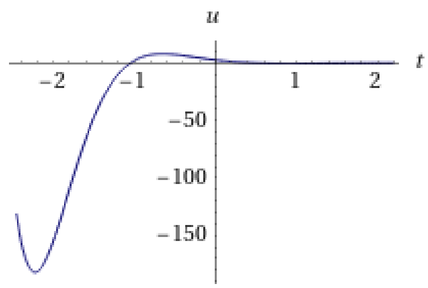
C. (*t* from –1.3 to 1.4)



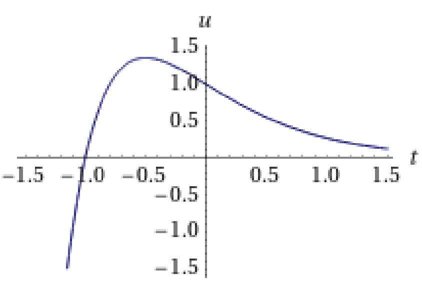
D. (*t* from –2.2 to 2.5)



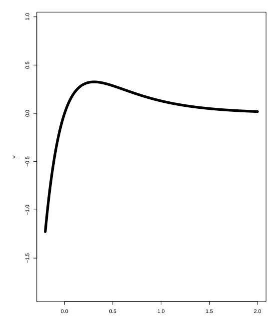
E. (*t* from –1.4 to 1.3)



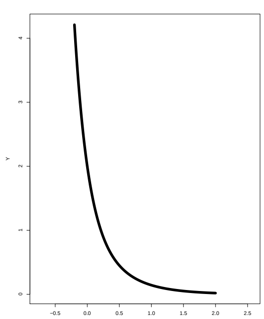
F. (*t* from –2.5 to 2.2)



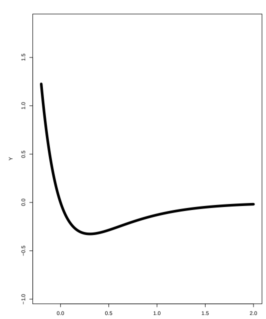
G. (*t* from –1.5 to 1.5)



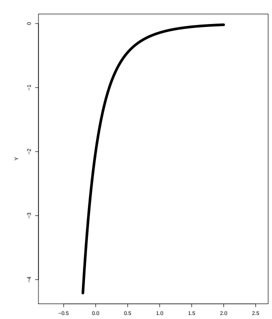
H.



I.



J.



K.