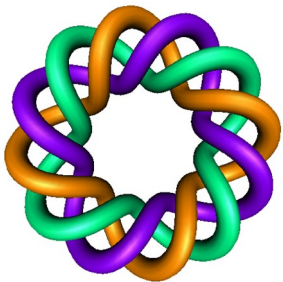


A very elementary introduction to proofs

Part 2

Example: Prove a function is not 1:1



By Dr. Isabel Darcy,
Dept of Mathematics and AMCS,
University of Iowa

Some notation:

\forall = for all

\exists = there exists

! = unique

$\exists!$ = there exists a unique

TFAE = The following are equivalent

$[p \Rightarrow q]$ is equivalent to $[\forall p, q \text{ holds}]$.

That is, for everything satisfying the hypothesis p , the conclusion q must hold.

$f : A \rightarrow B$ is 1:1 iff

$f(x_1) = f(x_2)$ implies $x_1 = x_2$.

$f : A \rightarrow B$ is 1:1 iff

$\forall x_1$ and $\forall x_2$ such that $f(x_1) = f(x_2)$,

we have $x_1 = x_2$.

How do we prove a function is **NOT** 1:1

A statement is **false** if the hypothesis holds, but the conclusion need not hold.

TFAE (The following are equivalent):

Hypothesis does not implies conclusion.

p does not imply q .

$p \not\Rightarrow q$.

It is not true that $\forall p, q$ holds.

$\exists p, q$ holds.

That is there exists **a specific case** where the hypothesis holds, but the conclusion does not hold.

$\sim [p \Rightarrow q]$ is equivalent to $\sim [\forall p, q \text{ holds}]$.

Thus if $p \Rightarrow q$ is false,

then it is not true that $[\forall p, q \text{ holds}]$.

That is, $\exists p$ such that q does not hold.

To prove that a statement is false:

Find an example where the hypothesis holds, but the conclusion does not hold.

$f : A \rightarrow B$ is 1:1 iff

$\forall x_1$ and $\forall x_2$ such that $f(x_1) = f(x_2)$,
we have $x_1 = x_2$.

Ex: To prove a function is not 1:1, find specific x_1, x_2 such that $f(x_1) = f(x_2)$, but $x_1 \neq x_2$.

Ex: To prove a function is not 1:1, find specific x_1, x_2 such that $f(x_1) = f(x_2)$, but $x_1 \neq x_2$.

Ex: $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$ is not 1:1

Proof:

$$f(1) = 1^2 = 1 = (-1)^2 = f(-1), \text{ but } 1 \neq -1$$

Contrapositive of $[p \implies q]$ is $[\sim q \implies \sim p]$.

Example: The contrapositive of

$f(x_1) = f(x_2)$ implies $x_1 = x_2$.
is

$x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

Example: The contrapositive of

$\ln(x_1) = \ln(x_2)$ implies $x_1 = x_2$.
is

$x_1 \neq x_2$ implies $\ln(x_1) \neq \ln(x_2)$.

The contrapositive of a theorem is true:

If $p \Rightarrow q$ is true, then

its contrapositive $\sim q \Rightarrow \sim p$ is also true.

If the conclusion q does not hold,

then the hypothesis p cannot hold.

If $x_1 \neq x_2$ then $\ln(x_1) \neq \ln(x_2)$

since $f(x) = \ln(x)$ is 1:1.

Sidenote: The *converse* of $[p \Rightarrow q]$ is $[q \Rightarrow p]$.

The converse of a theorem need not be true.

That is $p \Rightarrow q$ does not imply $q \Rightarrow p$.

TFAE:

- ▶ $f : A \rightarrow B$ is 1:1.
- ▶ $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- ▶ $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.
- ▶ $\forall x_1 \neq x_2, f(x_1) \neq f(x_2)$.

$f : A \rightarrow B$ is NOT 1:1 iff

$$\exists x_1 \neq x_2 \text{ such that } f(x_1) = f(x_2).$$