A very elementary introduction to proofs

Part 2

Example: Prove a function is not 1:1

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Some notation:

$\forall$ = for all

$\exists$ = there exists

$!$ = unique

$\exists\,!$ = there exists a unique

TFAE = The following are equivalent

$[p \Rightarrow q]$ is equivalent to $[\forall p,\ q$ holds].

That is, for everything satisfying the hypothesis p, the conclusion q must hold.

$f : A \to B$ is 1:1 iff

$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2.$$

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$\forall x_1$ and $\forall x_2$ such that $f(x_1) = f(x_2)$,

we have $x_1 = x_2$.

How do we prove a function is **NOT** 1:1

A statement is **false** if the hypothesis holds, but the conclusion need not hold.

TFAE (The following are equivalent):

Hypothesis does not implies conclusion.

p does not imply q.

$$p \not\Rightarrow q.$$

It is not true that $\forall p$, q holds.

$\exists p$, q holds.

That is there exists **a specific case** where the hypothesis holds, but the conclusion does not hold.

$\sim [p \Rightarrow q]$ is equivalent to $\sim [\forall p,\ q$ holds$]$.

Thus if $p \Rightarrow q$ is false,

then it is not true that $[\forall p,\ q$ holds$]$.

That is, $\exists p$ such that q does not hold.

To prove that a statement is false:

Find an example where the hypothesis holds, but the conclusion does not hold.

$f : A \to B$ is 1:1 iff

$\forall x_1$ and $\forall x_2$ such that $f(x_1) = f(x_2)$,

we have $x_1 = x_2$.

Ex: To prove a function is not 1:1, find specific x_1, x_2 such that $f(x_1) = f(x_2)$, but $x_1 \neq x_2$.

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Ex: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not 1:1

Proof:

$f(1) = 1^2 = 1 = (-1)^2 = f(-1)$, but $1 \neq -1$

Contrapositive of $[p \implies q]$ is $[\sim q \implies \sim p]$.

Example: The contrapositive of

$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2.$$

is

$$x_1 \neq x_2 \text{ implies } f(x_1) \neq f(x_2).$$

Example: The contrapositive of

$$ln(x_1) = ln(x_2) \text{ implies } x_1 = x_2.$$

is

$$x_1 \neq x_2 \text{ implies } ln(x_1) \neq ln(x_2).$$

The contrapositive of a theorem is true:

If $p \Rightarrow q$ is true, then

its contrapostive $\sim q \Rightarrow \sim p$ is also true.

If the conclusion q does not hold,

then the hypothesis p cannot hold.

If $x_1 \neq x_2$ then $ln(x_1) \neq ln(x_2)$

since $f(x) = ln(x)$ is 1:1.

Sidenote: The *converse* of $[p \implies q]$ is $[q \implies p]$.

The converse of a theorem need not be true.

That is $p \Rightarrow q$ does not imply $q \Rightarrow p$.

TFAE:

- $f : A \to B$ is 1:1.
- $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.
- $\forall x_1 \neq x_2,\ f(x_1) \neq f(x_2)$.

$f : A \to B$ is NOT 1:1 iff

$$\exists x_1 \neq x_2 \text{ such that } f(x_1) = f(x_2).$$