EXISTANCE

$span 1, t, t^2 =$ polynomials of degree 2

A polynomial $p(t)$ is in the span of $\{1, t, t^2\}$ if and only if there exists a solution for a, b, c to the equation

$$p(t) = a + bt + ct^2$$

Example 1: $2 + t^3$ is not in the span of $\{1, t, t^2\}$ since there does not exist a, b, c such that $2+t^3 = a+bt+ct^2$

Example 2: $1+2t+3t^2$ is in the span of $\{1, t, t^2\}$ since there exists a, b, c such that $1 + 2t + 3t^2 = a + bt + ct^2$

$a = 1, b = 2, c = 3$ is a solution.

UNIQUENESS

$\mathbf{b_1}, ..., \mathbf{b_n}$ are linearly independent if

$$c_1\mathbf{b_1} + c_2\mathbf{b_2} + ... + c_n\mathbf{b_n} = d_1\mathbf{b_1} + d_2\mathbf{b_2} + ... + d_n\mathbf{b_n}$$
$$\text{implies } c_1 = d_1,\ c_2 = d_2 ...,\ c_n = d_n.$$

or equivalently,

$\mathbf{b_1}, ..., \mathbf{b_n}$ are linearly independent if

$c_1\mathbf{b_1} + c_2\mathbf{b_2} + ... + c_n\mathbf{b_n} = 0$ implies $c_1 = c_2 = ...c_n$.

Example 1: $\mathbf{b_1} = (1,0,0)$, $\mathbf{b_2} = (0,1,0)$, $\mathbf{b_3} = (0,0,1)$. $\blacksquare$

$$(1,2,3) \neq (1,2,4).$$

If $(a,b,c) = (1,2,3)$ then $a = 1,\ b = 2,\ c = 3$.

Example 2: $\mathbf{b_1} = 1$, $\mathbf{b_2} = t$, $\mathbf{b_3} = t^2$.

$$1 + 2t + 3t^2 \neq 1 + 2t + 4t^2.$$

If $a + bt + ct^2 = 1 + 2t + 3t^2$ then $a = 1,\ b = 2,\ c = 3$.

Application: Partial Fractions

$$\frac{4}{(x^2+1)(x-3)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-3}$$

If you don't like denominators, get rid of them:

$$4 = (Ax + B)(x - 3) + C(x^2 + 1)$$

$$4 = Ax^2 + Bx - 3Ax - 3B + Cx^2 + C$$

$$4 = (A + C)x^2 + (B - 3A)x - 3B + C$$

I.e., $0x^2+0x+4 = (A + C)x^2 + (B - 3A)x - 3B + C$

Thus $0 = A + C, \quad 0 = B - 3A, \qquad 4 = -3B + C.$

$C = -A, \;\; B = 3A, \;\; 4 = -3(3A) + -A \Rightarrow 4 = -10A.$

Hence $A = -\frac{2}{5}$, $B = 3(-\frac{2}{5}) = -\frac{6}{5}$, $C = \frac{2}{5}$.

$$\text{Thus, } \frac{4}{(x^2+1)(x-3)} = \frac{-\frac{2}{5}x-\frac{6}{5}}{x^2+1} + \frac{\frac{2}{5}}{x-3}$$

$$= \frac{-2x-6}{5(x^2+1)} + \frac{2}{5(x-3)}$$

Alternatively, can plug in $x = 3$ to quickly find C and then solve for A and B. Can also use matrices to solve linear eqns.