A Quick Review of

Linear Algebra

(linear combination, linear independence, span, basis)

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Partial Fractions

for

Differential Equations

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LINEAR COMBINATION

$\mathbf{p}$ is a linear combination of $\{\mathbf{b_1}, \mathbf{b_2}, \cdots, \mathbf{b_n}\}$ iff there exists c_i such that

$c_i \in \mathbb{R}$

$$\mathbf{p} = c_1\mathbf{b_1} + c_2\mathbf{b_2} + \ldots + c_n\mathbf{b_n}$$

Example 1:
Let $\mathbf{b_1} = (1, 0, 0)$, $\mathbf{b_2} = (0, 1, 0)$, $\mathbf{b_3} = (0, 0, 1)$.

$(1, 2, 3)$ is linear combination of

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

since $\boxed{(1, 2, 3) = 1((1, 0, 0) + 2(0, 1, 0) + 3(0, 0, 1)}$

LINEAR COMBINATION

$\mathbf{p}$ is a linear combination of $\{\mathbf{b_1}, \mathbf{b_2}, \cdots, \mathbf{b_n}\}$ iff there exists c_i such that

$$\mathbf{p} = c_1\mathbf{b_1} + c_2\mathbf{b_2} + ... + c_n\mathbf{b_n}$$

Example 2: Let $\mathbf{b_1} = 1$, $\mathbf{b_2} = t$, $\mathbf{b_3} = t^2$

Then $1 + 2t + 3t^2$ is a linear combination of $\{1, t, t^2\}$

Sidenote: $(1, 2, 3)$ can be used to represent the polynomial $1 + 2t + 3t^2$.

Sidenote = we won't need this for this class.

EXISTENCE

$\mathbf{p}$ is in $span\{\mathbf{b_1}, \mathbf{b_2}, \cdots, \mathbf{b_n}\}$ iff there **exists** c_i such that

$$\boxed{\mathbf{p} = c_1\mathbf{b_1} + c_2\mathbf{b_2} + \ldots + c_n\mathbf{b_n}}$$

Example: $span\{1, t, t^2\}$ = polynomials of degree at most 2.

A polynomial $p(t)$ is in the span of $\{1, t, t^2\}$ if and only if there **exists** a solution for a, b, c to the equation

$$p(t) = a + bt + ct^2$$

at least one soln.

EXISTENCE

at least one sol'n

Example 1: $2+t^3$ is not in the span of $\{1, t, t^2\}$ since there does not exist a, b, c such that

$$2+t^3 = a+bt+ct^2$$

Example 2: $1+2t+3t^2$ is in the span of $\{1, t, t^2\}$ since there exists a, b, c such that

$$1+2t+3t^2 = a+bt+ct^2$$

In particular, $a=1, b=2, c=3$ is a solution.

UNIQUENESS

at most one sol'n

$\mathbf{b_1}, ..., \mathbf{b_n}$ are linearly independent iff

$$c_1\mathbf{b_1} + c_2\mathbf{b_2} + ... + c_n\mathbf{b_n} = 0 \implies c_1 = ... = c_n = 0$$

or equivalently,

$\mathbf{b_1}, ..., \mathbf{b_n}$ are linearly independent iff

$$c_1\mathbf{b_1} + c_2\mathbf{b_2} + ... + c_n\mathbf{b_n} = d_1\mathbf{b_1} + d_2\mathbf{b_2} + ... + d_n\mathbf{b_n}$$

$$\implies c_1 = d_1, c_2 = d_2, ... = c_n = d_n.$$

In other words, if a solution exists for the following equation, then the solution is **unique**:

$$\boxed{\mathbf{p}} = c_1\mathbf{b_1} + c_2\mathbf{b_2} + ... + c_n\mathbf{b_n}$$

unique representive if p is in span

UNIQUENESS

Example 1:
$\mathbf{b_1} = (1, 0, 0)$, $\mathbf{b_2} = (0, 1, 0)$, $\mathbf{b_3} = (0, 0, 1)$.

$$(1, 2, 3) \neq (1, 2, 4).$$

If $(a, b, c) = (1, 2, 3)$, then $a = 1$, $b = 2$, $c = 3$.

Example 2: $\mathbf{b_1} = 1$, $\mathbf{b_2} = t$, $\mathbf{b_3} = t^2$.

$$1 + 2t + 3t^2 \neq 1 + 2t + 4t^2.$$

If $a + bt + ct^2 = 1 + 2t + 3t^2$, then $a = 1$, $b = 2$, $c = 3$.

BASIS

$\{\mathbf{b_1}, \mathbf{b_2}, \cdots, \mathbf{b_n}\}$ is a basis for the vector space V if

1.) $span\{\mathbf{b_1}, \mathbf{b_2}, \cdots, \mathbf{b_n}\} = V$ and

2.) $\{\mathbf{b_1}, \mathbf{b_2}, \cdots, \mathbf{b_n}\}$ is a linearly independent set.

In other words if $\mathbf{p} \in V$, then there **exists** a solution for c_i for the following equation and that solution is **unique**:

$$\mathbf{p} = c_1\mathbf{b_1} + c_2\mathbf{b_2} + ... + c_n\mathbf{b_n}$$

Example 1: $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis for $\mathbb{R}^3$.

Example 2: $\{1, t, t^2\}$ = is a basis for the set of polynomials of degree at most 2.

Application: Partial Fractions

① Factor over $\mathbb{R}$

$$\underbrace{\frac{x+1}{(x+2)(x-3)}}_{\text{LHS}} = \underbrace{\frac{A}{x+2} + \frac{B}{x-3}}_{\text{RHS}}$$

deg 1 deg 0 deg 2

$$\frac{4}{(x^2+1)(x-3)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-3} + \frac{Dx^2+Ex+F}{x^3+x+1}$$

(x^3+x+1)

~~$(x+i)(x-i)$~~

deg 2 deg 1 deg 3

deg 0

$$\frac{x^4+1}{(x^2+1)(x-3)^3} = \frac{Ax+B}{x^2+1} + \frac{C}{(x-3)^1} + \frac{D}{(x-3)^2} + \frac{E}{(x-3)^3}$$

deg 1 deg 1

$$\frac{4}{(x^2+1)^2(x-3)} = \frac{Ax+B}{(x^2+1)^1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x-3}$$

deg 2

$x-1 = x+\underbrace{1-1}_{+\text{"}0\text{"}}-1$

Don't forget to simplify first

both are deg = 1

$$\frac{x^2-1}{(x+1)^2} = \frac{(x-1)\cancel{(x+1)}}{(x+1)^{\cancel{2}}} = \boxed{\frac{x-1}{x+1}} = \frac{x+1-1-1}{x+1}$$

$$= \frac{x+1-2}{x+1} = \frac{x+1}{x+1} + \frac{-2}{x+1} = \boxed{1+\frac{-2}{x+1}}$$

For partial fractions, the power in numerator must be less than the power in denominator.

If power in numerator $\boxed{\geq}$ power in denominator, do long division first (or add a "0" and simplify algebraically).

Solve for A, B, C

Application: Partial Fractions

$$\cancel{(x^2+1)(x-3)}\left[\frac{4}{\cancel{(x^2+1)(x-3)}}\right] = \left[\frac{Ax+B}{x^2+1} + \frac{C}{x-3}\right](x^2+1)(x-3)$$

If you don't like denominators, get rid of them:

$$\boxed{4 = (Ax+B)(x-3) + C(x^2+1)}$$

$$4 = Ax^2 + Bx - 3Ax - 3B + Cx^2 + C$$

combine like term

$$\boxed{4} = \boxed{(A+C)x^2 + (B-3A)x - 3B + C}$$

↓ linear algebra

I.e.,

$$0x^2 + 0x + 4 = (A+C)x^2 + (B-3A)x - 3B + C$$

$\{x^2, x, 1\}$ is lin indep

$$\boxed{0}x^2 + 0x + 4 = \boxed{(A+C)}x^2 + (B - 3A)x - 3B + C$$

Thus $\boxed{0 = A + C}$, $\quad 0 = B - 3A$, $\quad 4 = -3B + C$

$C = -A$, $\;B = 3A$, $\qquad 4 = -3(3A) + -A \Rightarrow$
$4 = -10A$.

Hence $A = -\frac{2}{5}$, $\quad B = 3(-\frac{2}{5}) = -\frac{6}{5}$, $\quad C = \frac{2}{5}$.

Thus, $\frac{4}{(x^2+1)(x-3)} = \frac{-\frac{2}{5}x - \frac{6}{5}}{x^2+1} + \frac{\frac{2}{5}}{x-3}$

$$= \boxed{\frac{-2x-6}{5(x^2+1)} + \frac{2}{5(x-3)}}$$

Note there are many correct ways to solve for A, B, C. For example, one can plug in $x = 3$ to quickly find C and then solve for A and B.

$$\boxed{4 = (Ax + B)(x - 3) + C(x^2 + 1)}$$

One can also use matrices to solve linear eqns.

Let $x = 3$: $4 = \cancel{\sim (3-3)} + c(9+1)$

$\Rightarrow c = \frac{4}{10} = \frac{2}{5}$

Let $x = 0$: $(0+B)(-3) + c(1) = 4$