Note that $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$ and $A(c\mathbf{x}) = cA\mathbf{x}$

Linearity is the best thing ever

A system of equations is $A\mathbf{x} = \mathbf{b}$ is **homogeneous** if $\mathbf{b} = \mathbf{0}$.

Linear

MATH 2700

Suppose $A\underline{\mathbf{u}} = \mathbf{0}$, $A\underline{\mathbf{v}} = \mathbf{0}$, and $A\underline{\mathbf{p}} = \underline{\mathbf{b}}$, then

$$\begin{aligned} A(c_1\underline{\mathbf{u}} + c_2\underline{\mathbf{v}} + \underline{\mathbf{p}}) &= \cancel{c_1 A\mathbf{u}}^{=0} + \cancel{c_2 A\mathbf{v}}^{=0} + \underline{A\mathbf{p}} \\ &= c_1(\mathbf{0}) + c_2(\mathbf{0}) + \mathbf{b} = \underline{\mathbf{b}} \end{aligned}$$

I.e., $\mathbf{x} = \underbrace{c_1\mathbf{u} + c_2\mathbf{v}}_{\text{general homog}} + \mathbf{p}$ is a soln to $A\mathbf{x} = \mathbf{b}$ for any c_1, c_2.

general homog $\quad$ ← one non homog

Solve the following systems of equations:

①

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

② $\qquad$ ③

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$\begin{array}{cccccc} & & & ① & ② & ③ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 0 & 2 \\ 4 & 5 & 6 & 0 & 3 & 5 \\ 7 & 8 & 9 & 0 & 0 & 8 \end{array}\right]$$

$\downarrow R_2 - 4R_1 \to R_2, \quad R_3 - 7R_1 \to R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 2 \\ 0 & -3 & -6 & 0 & 3 & -3 \\ 0 & -6 & -12 & -7 & 0 & -6 \end{bmatrix}$$

$\downarrow R_3 - 2R_1 \to R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 2 \\ 0 & -3 & -6 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 & -6 & 0 \end{bmatrix}$$

$\downarrow$ already know sol'n to system b.

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 2 \\ 0 & -3 & -6 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\downarrow -\frac{1}{3}R_2 \to R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 - 2R_2 \to R_1]{} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Note similarities & differences btw 2700 & 3600

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

no solution

compare to 3600.

If fns are continuous $\exists !$ soln to IVP

difference for 2700

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Check:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \& \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

Compare to solving linear homogeneous differential eqn:

Ex: $ay'' + by' + cy = g(t)$ 3600

1.) Easily solve homogeneous DE: $ay'' + by' + cy = 0$

$y = e^{rt} \Rightarrow ar^2 + br + c = 0 \Rightarrow y = c_1\phi_1 + c_2\phi_2$ for homogeneous solution (see sections 3.1, 3.3, 3.4).

2.) More work: Find one solution to $ay'' + by' + cy = g(t)$ (see sections 3.5, 3.6)

If $y = \psi(t)$ is a soln, then general soln to $ay''+by'+cy = g(t)$ is

$$\boxed{y = c_1\phi_1 + c_2\phi_2 + \psi}$$

because of linearity

Check: $a\phi_1'' + b\phi_1' + c\phi_1 = 0$

$a\phi_2'' + b\phi_2' + c\phi_2 = 0$

$a\psi'' + b\psi' + c\psi = g(t)$

To solve $ay'' + by' + cy = g_1(t) + g_2(t)$ $+ g_3(t)$

1.) Solve $ay'' + by' + cy = 0 \quad \Rightarrow \quad y = c_1\phi_1 + c_2\phi_2$ for homogeneous solution.

2a.) Solve $ay'' + by' + cy = g_1(t) \quad \Rightarrow \quad y = \psi_1$

2b.) Solve $ay'' + by' + cy = g_2(t) \quad \Rightarrow \quad y = \psi_2$

General solution to $ay'' + by' + cy = g_1(t) + g_2(t)$ is $+ g_3(t)$

$$y = c_1\phi_1 + c_2\phi_2 + \psi_1 + \psi_2 \; + \psi_3$$ etc