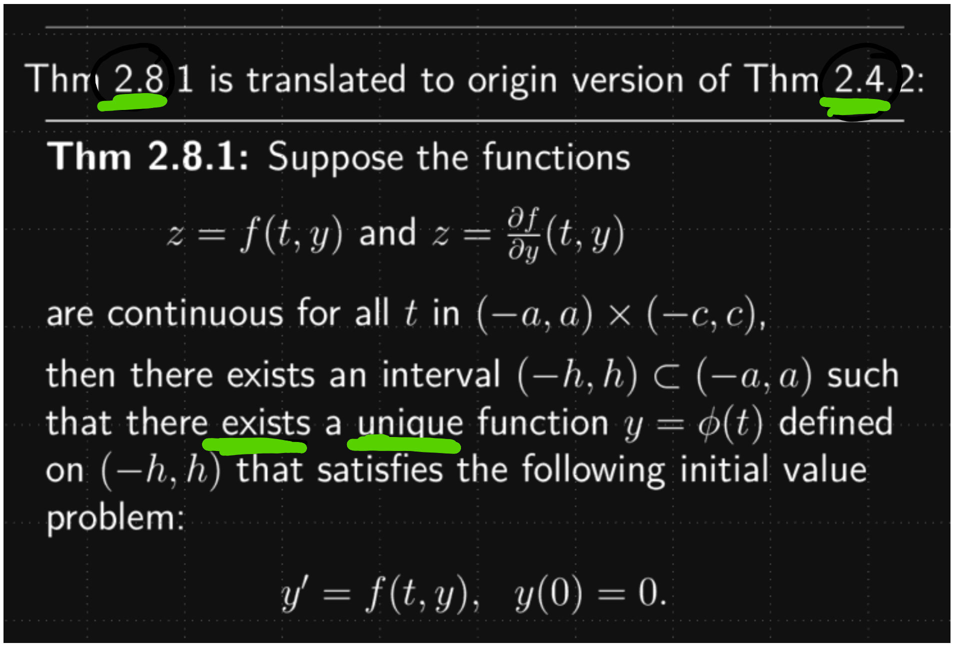
Monday, September 28, 2020 1:51 PM



Thm 2.8.1 is translated to origin version of Thm 2.4.2:

Thm 2.8.1: Suppose the functions

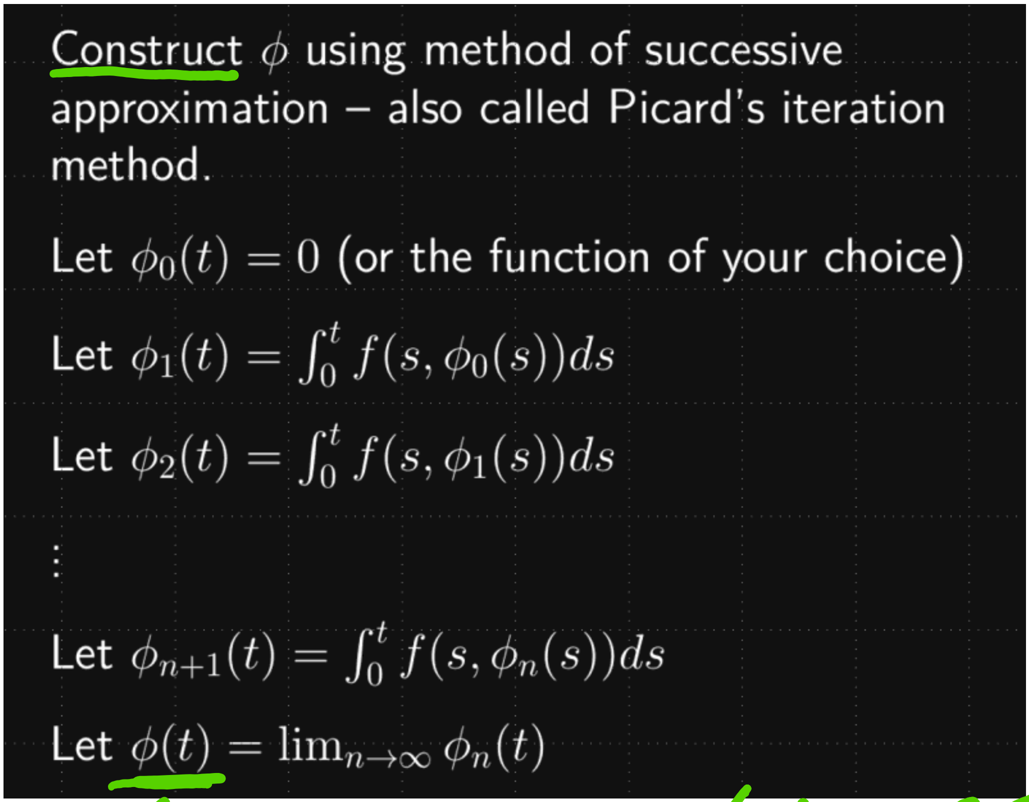
$$z = f(t, y) \text{ and } z = \frac{\partial f}{\partial y}(t, y)$$

are continuous for all t in $(-a, a) \times (-c, c)$,

then there exists an interval $(-h, h) \subset (-a, a)$ such that there exists a unique function $y = \phi(t)$ defined on $(-h, h)$ that satisfies the following initial value problem:

$$y' = f(t, y), \quad y(0) = 0.$$

Proof outline: Constructive proof



Construct ϕ using method of successive approximation – also called Picard's iteration method.

Let $\phi_0(t) = 0$ (or the function of your choice)

Let $\phi_1(t) = \int_0^t f(s, \phi_0(s))ds$

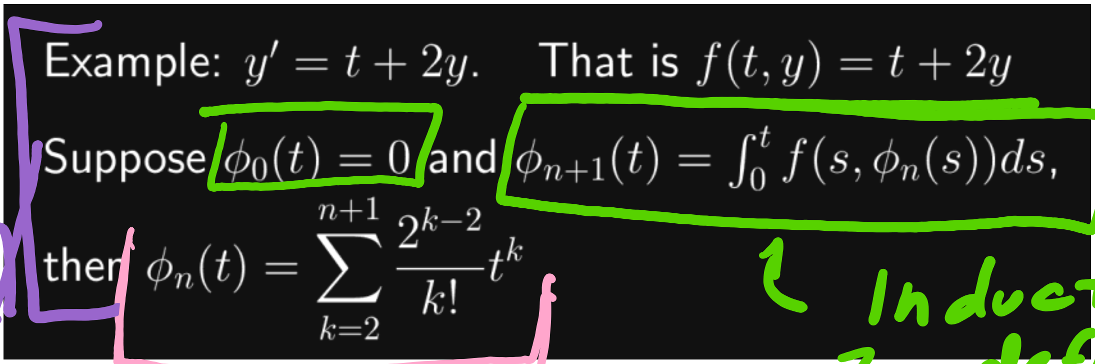
Let $\phi_2(t) = \int_0^t f(s, \phi_1(s))ds$

$\vdots$

Let $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds$

Let $\phi(t) = \lim_{n \to \infty} \phi_n(t)$

ℓ claim ϕ is soln to IVP



Example: $y' = t + 2y$. That is $f(t, y) = t + 2y$

Suppose $\phi_0(t) = 0$ and $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds$,

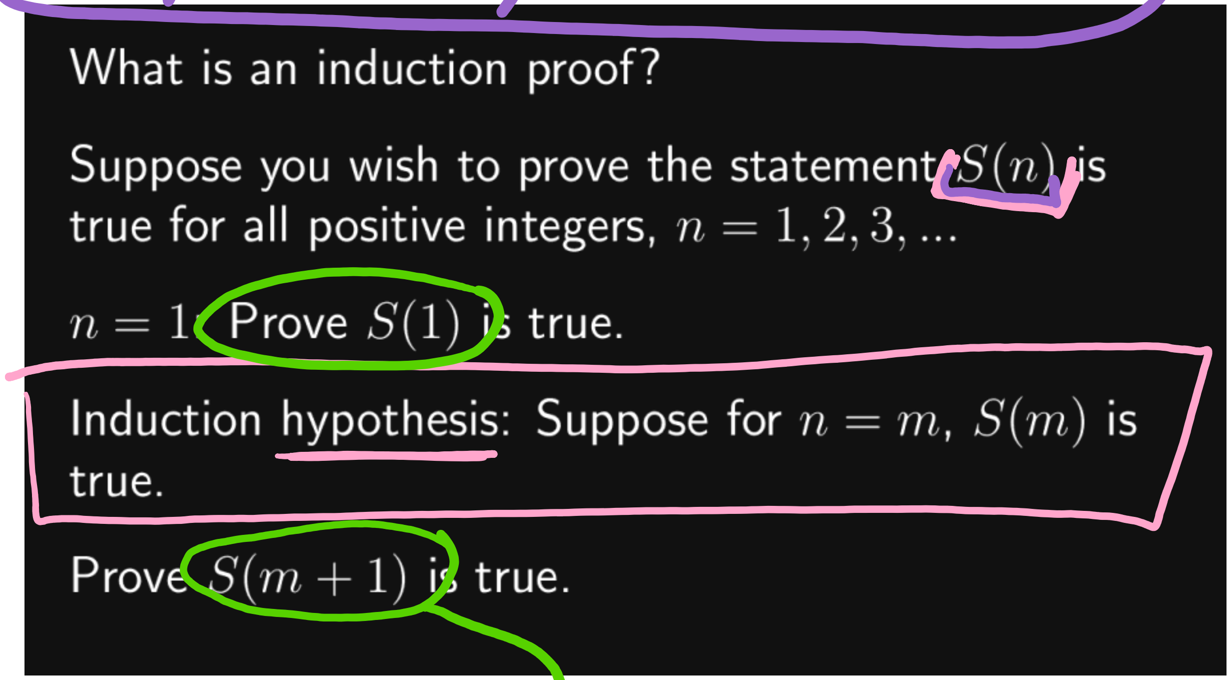
then $\phi_n(t) = \sum_{k=2}^{n+1} \frac{2^{k-2}}{k!} t^k$

Inductive defn = recursive defn

$\phi_0 \Rightarrow \phi_1 \Rightarrow \phi_2 \Rightarrow \dots$

found pattern for ϕ_n

proof by induction



What is an induction proof?

Suppose you wish to prove the statement $S(n)$ is true for all positive integers, $n = 1, 2, 3, \dots$

$n = 1$: Prove $S(1)$ is true.

Induction hypothesis: Suppose for $n = m$, $S(m)$ is true.

Prove $S(m + 1)$ is true.

$S(1)$ true $\Rightarrow$ $S(1+1)$ is true

$S(2)$ is true

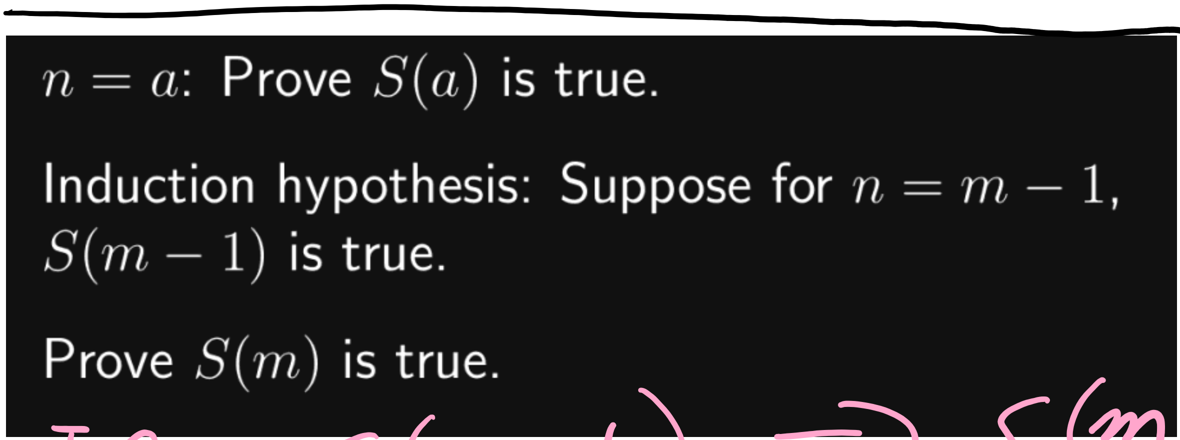
If $S(m)$ true $\Rightarrow$ $S(m+1)$ true

then $S(1) \Rightarrow S(2) \Rightarrow S(3) \Rightarrow \dots \Rightarrow S(n)$

true true true true

then $S(1) \Rightarrow S(2) \Rightarrow S(3) \Rightarrow \ldots$

true true true true for arbitrary



$n = a$: Prove $S(a)$ is true.

Induction hypothesis: Suppose for $n = m - 1$, $S(m-1)$ is true.

Prove $S(m)$ is true.

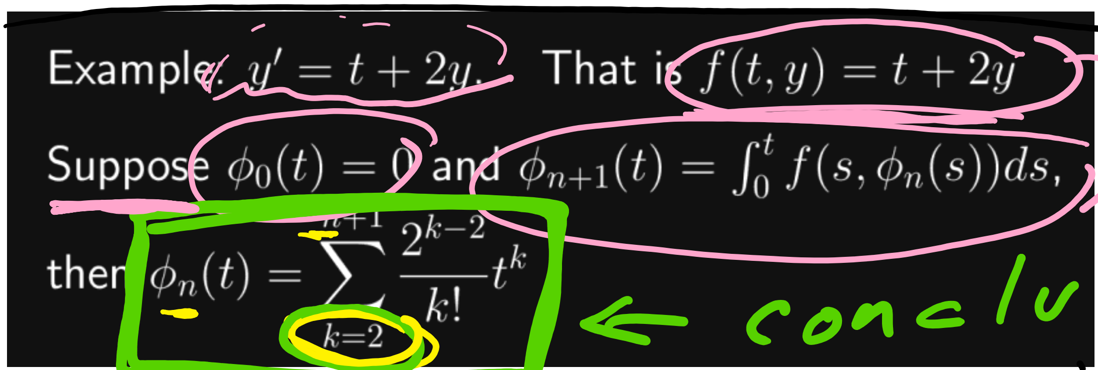
If $S(m-1) \Rightarrow S(m)$

$S(a) \Rightarrow S(a+1) \Rightarrow S(a+2) \Rightarrow \ldots$

$\Rightarrow S(n) \Rightarrow$

$a = m-1 \Rightarrow a+1 = m$

true for arbitrary n



Example: $y' = t + 2y$. That is $f(t, y) = t + 2y$

Suppose $\phi_0(t) = 0$ and $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds$,

then $\phi_n(t) = \sum_{k=2}^{n+1} \frac{2^{k-2}}{k!} t^k$

hypothis

← conclusion

Proof by induction

We are given

✓ ① $f(t,y) = t+2y$

✓ ② $\phi_0(t) = 0$

✓ ③ $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds$

We won't need

→ (1) $y' = t+2y$

this since

we have ①

$$\boxed{\text{Claim: } \underbrace{\phi_n(t) = \sum_{k=2}^{n+1} \frac{2^{k-2}}{k!} t^k}}$$

(do not assume what you are trying to prove (unless it is your induction hypothesis)

So can't use formule to prove this n=1, but want to prove this formula holds for n=1

Claim: for $n=1$ $\underbrace{\phi_1(t) = \sum_{k=2}^{2} \frac{2^{k-2}}{k!} t^k}$

prove this

$$\text{RHS: } \sum_{k=2}^{2} \frac{2^{k-2}}{k!} t^k = \frac{2^0}{2!} t^2 = \frac{t^2}{2!}$$

$$\text{LHS: } \phi_1(t) = \int_0^t f(s, \phi_0(s))\,ds$$

$$= \int_0^t f(s, 0)\,ds = \int_0^t (s + 2(0))\,ds = \frac{s^2}{2}\Big|_0^t \boxed{= \frac{t^2}{2}}$$

by hypothesis ① $f(t, y) = t + 2y$

② $\phi_0(t) = 0$

③ $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\,ds$

Thus $\phi_1(t) = \frac{t^2}{2} = \frac{t^2}{t!} = \sum_{k=2}^{2} \frac{2^{k-2}}{k!} t^k$

Induction hypothesis

Suppose for $n = m$

$$\phi_m(t) = \sum_{k=2}^{m+1} \frac{2^{k-2}}{k!} t^k$$

Claim true for $n = m+1$

$$\boxed{\text{Claim: } \phi_{m+1}(t) = \sum_{k=2}^{m+2} \frac{2^{k-2}}{k!} t^k}$$

Know: ① $f(t,y) = t+2y$ ② $\phi_0(t) = 0$

③ $\phi_{m+1}(t) = \int_0^t f(s, \phi_m(s))\,ds$

④ Induction hyp: $\phi_m(t) = \sum_{k=2}^{m+1} \frac{2^{k-2}}{k!} t^k$

LHS

$(t) = \int^t f(s, \phi(s))\,ds$

LHS

$$\underline{\phi_{m+1}}(t) \underset{③}{=} \int_0^t f(\underline{s}, \underline{\phi_m}(s))\,ds$$

$$\underset{①}{=} \int_0^t \left(s + 2\underline{\phi_m}(s)\right) ds$$

by induction hypothesis ⇒ ④

$$\underset{④}{=} \int_0^t \left(s + 2\sum_{k=2}^{m+1} \frac{2^{k-2}}{k!} s^k\right) ds$$

$$= \int_0^t s\,ds + 2\sum_{k=2}^{m+1} \frac{2^{k-2}}{k!} \int_0^t s^k\,ds$$

$$= \frac{s^2}{2}\Big|_0^t + 2\sum_{k=2}^{m+1} \frac{2^{k-2}}{k!} \frac{s^{k+1}}{k+1}\Big|_0^t$$

$$= \frac{t^2}{2} + 2\sum_{k=2}^{m+1} \frac{2^{k-2}}{k!} \frac{t^{k+1}}{k+1} - 0$$

lgebraic simplificat

$$\boxed{= \frac{t^2}{2} + \sum_{k=2}^{m+1} \frac{2\cdot 2^{k-2}}{k!} \frac{t^{k+1}}{k+1}}$$

Goal:

$$\sum_{k=2}^{m+2} \frac{2^{k-2}}{k!} t^k$$

alge

$k=2 \quad \ldots \quad k+1$

$k=2 \quad k!$

$$= \frac{t^2}{2} + \sum_{k=2}^{m+1} \frac{2^{k-1}}{(k+1)!} t^{k+1}$$

note $\frac{t^2}{2} = \frac{2^{1-1} t^{1+1}}{(1+1)!}$ ie $k=1$ term

$$= \sum_{k=1}^{m+1} \frac{2^{k-1}}{(k+1)!} t^{k+1}$$

$m+1$ terms

$$= \sum_{k=2}^{m+2} \underbrace{\frac{2^{k-2}}{k!} t^{k}} \quad \blacksquare$$

K is increased by 2

K needs to be decrcase by 1