Exam 1 Wednesday over 2.1, 2.4, 2.5, 2.8 (including induction), ch 3, 4, 7.1 – 7.5

90 minutes

1:30 – 4

start before 2:31 pm ← end

and 5 pm – 6:30 pm ???

← end

stability

7.2: 17, 19 ← check answer

7.3: 13, 14, 15 ← e. values / vector

7.1 (use matrix form): 3, 4, 5, 6, 12

7.5: 1b, 2b, 5b ← solve

similar to HW

not 7.4

Problem session Monday during class. ← ask question

Problem session Tuesday ??? (submit survey by Sunday night if you wish to help choose the time).

HW 8, ungraded survey due Sunday night.

3 pts

Real quiz 3 due Monday night.

In class: All Quizzes 10/7 - 10/21 due Monday night. ← Bonus pts for in class quizzes

HW 9 knowledge due Wednesday (but can turn in Friday – Sunday)

Quiz Instructions

Quiz 3

WHILE TAKING THIS QUIZ: Post links to any web resources that you use for this quiz to the pinned ICON discussion for Quiz 3. Note you should post the full URL.

open web

For example, if you use WolframAlpha to compute 1+1:

Incorrect post: https://www.wolframalpha.com/

Correct Post: https://www.wolframalpha.com/input/?i=1%2B1

Please include the problem #.

If the site requires payment and/or registration, please state so.

Note: you have an unlimited number of attempts.

If on write both sides don't

single side your scans please

for exam 2: send me a chat when you are done

$$\text{Solve } \mathbf{x}' = \begin{pmatrix} 3 & 1 \\ 5 & 0 \end{pmatrix} \mathbf{x}$$

Guessed $\vec{x} = \vec{v}e^{rt}$

r = e. value

w/ e. vector $\vec{v}$

Step 1: Find eigenvalues:

$$\begin{vmatrix} 3-r & 1 \\ 5 & 0-r \end{vmatrix} = (3-r)(-r) - 5 = r^2 - 3r - 5 = 0$$

$$\text{Thus } r = \frac{3 \pm \sqrt{9-4(1)(-5)}}{2} = \frac{3 \pm \sqrt{29}}{2}$$

Step 2: Find eigenvectors:

non zero solns to

$$\begin{pmatrix} 3 - (\frac{3\pm\sqrt{29}}{2}) & 1 \\ 5 & 0 - (\frac{3\pm\sqrt{29}}{2}) \end{pmatrix} \mathbf{v} = \mathbf{0}$$

Eigenvalues: $r = \frac{3 \mp \sqrt{29}}{2}$ $\boxed{7.5}$ $\boxed{\text{2 real e. vector case}}$

$$\left(\begin{matrix} \frac{3 \mp \sqrt{29}}{2} & 1 \\ 5 & \frac{-3 \mp \sqrt{29}}{2} \end{matrix} \right) \underbrace{\begin{bmatrix} -1 \\ \frac{3 \mp \sqrt{29}}{2} \end{bmatrix}}_{\text{e. vectors}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

General solution:

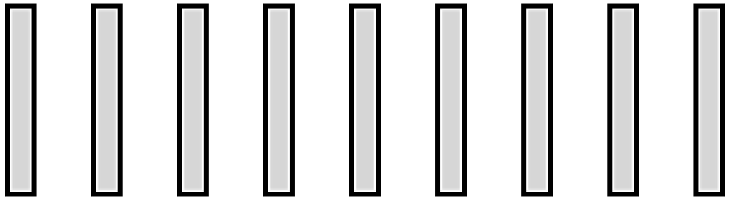
$$\boxed{\vec{x} = c_1 \left(\begin{bmatrix} -1 \\ \frac{3-\sqrt{29}}{2} \end{bmatrix} e^{\left(\frac{3+\sqrt{29}}{2} t\right)} \right) + c_2 \left(\begin{bmatrix} -1 \\ \frac{3+\sqrt{29}}{2} \end{bmatrix} e^{\left(\frac{3-\sqrt{29}}{2} t\right)} \right)}$$

Eigenvalues: $r = \frac{3 \pm \sqrt{29}}{2}$

$$\begin{pmatrix} \frac{3 \mp \sqrt{29}}{2}) & 1 \\ 5 & \frac{-3 \mp \sqrt{29}}{2}) \end{pmatrix} \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

General solution:

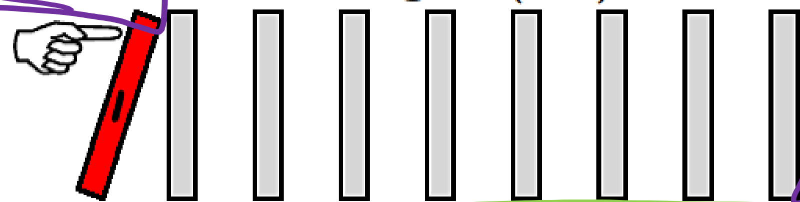
Domino effect of Induction



We have a statement $S(n)$ that we need to prove.

$\forall$ integers $\geq n_0$

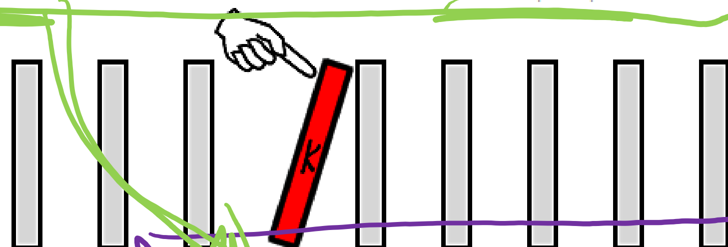
Base case: Proving $S(n_0)$ is like ~~knocking down the first domino in the~~ row:



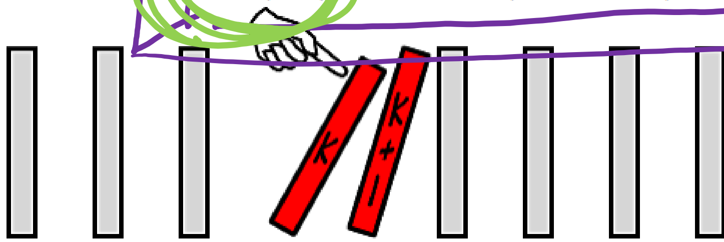
Show $S(n_0)$ is true.

2,8 induction

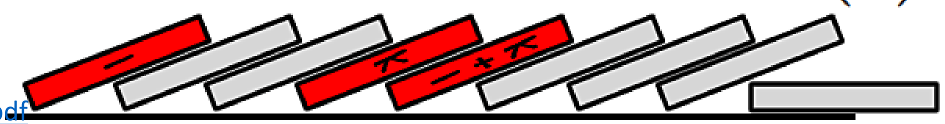
Hypothesis: Assume for some $k \geq n_0$, $S(k)$ holds



Induction Step: Show that $\boxed{S(k) \Rightarrow S(k+1)}$



If we do all of the above, all the dominoes fall: $S(n)$ holds!

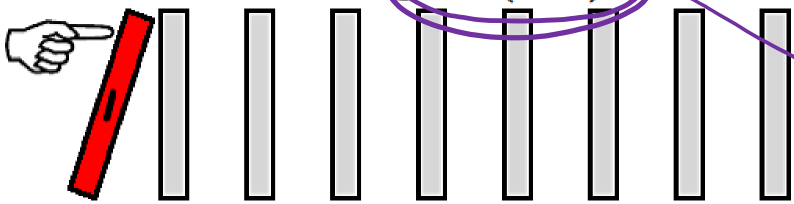


$\square$

pics courtesy: Coolmath Algebra

http://www.few.vu.nl/~rbakhshi/teaching/induction-handout.pdf

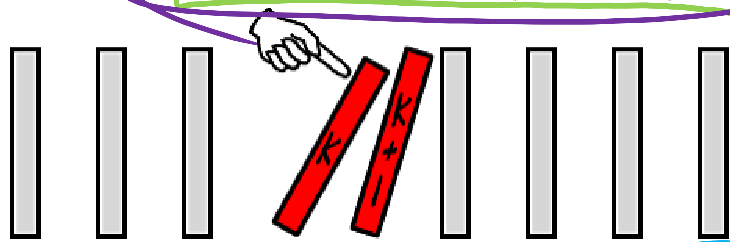
BASE CASE: Proving $S(n_0)$ is like knocking down the first domino in the row:



Show $S(n_0)$ is true.

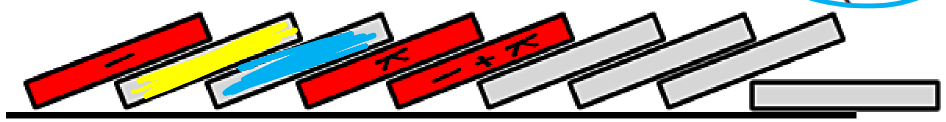
HYPOTHESIS: Assume for some $k \geq n_0$, $S(k)$ holds

INDUCTION STEP: Show that $S(k) \Rightarrow S(k+1)$



Prove these 2 in order to knock down all the dominoes

If we do all of the above, all the dominoes fall: $S(n)$ holds!



pics courtesy: Coolmath Algebra

$S(n_0) \Rightarrow S(n_0+1) \Rightarrow S(n_0+2) \Rightarrow \dots \Rightarrow S(j)$

Show by induction that if

2.8 $y' = f(t, y)$

approx

given

$$f(t, y) = 2t(1 + y),$$

2.8

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds,$$

given on exam 2

and $\phi_0(t) = 0$,

then $\phi_n(t) = \sum_{k=1}^{n} \frac{t^{2k}}{k!}$ for $n \geq 1$

given

Show by induction that if $f(t,y) = 2t(1+y)$,

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\,ds, \text{ and } \phi_0(t) = 0,$$

then $\phi_{\cancel{n}}(t) = \sum_{k=1}^{\cancel{n}} \frac{t^{2k}}{k!}$ for $n \geq 1$

Proof by induction on n:

Base case $n = 1$

Claim: $\phi_1(t) \stackrel{?}{=} \sum_{k=1}^{1} \frac{t^{2k}}{k!}$

Do not assume what you are trying to prove

Show by induction that if $f(t,y) = 2t(1+y)$, why

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds, \text{ and } \phi_0(t) = 0,$$

then $\phi_n(t) = \sum_{k=1}^{n} \frac{t^{2k}}{k!}$ for $n \geq 1$

LHS RHS

Proof by induction on n:

$n = 1$: $\phi_1(t) = \int_0^t f(s, \phi_0(s))ds$

$= \int_0^t f(s, 0)ds = \int_0^t 2s(1+0)ds$

$= \int_0^t 2s\,ds = s^2 \Big|_0^t = t^2$

Show by induction that if $f(t,y) = 2t(1+y)$,

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds, \text{ and } \phi_0(t) = 0,$$

then $\phi_n(t) = \displaystyle\sum_{k=1}^{n} \frac{t^{2k}}{k!}$ for $n \geq 1$

Proof by induction on n:

For $n = 1$,

LHS $\phi_1(t) = \int_0^t f(s, \phi_0(s))ds = \int_0^t f(s, 0)ds$

$$= \int_0^t 2s(1+0)ds = s^2|_0^t = t^2 \checkmark$$

RHS $\displaystyle\sum_{K=1}^{1} \frac{t^{2K}}{K!} = \frac{t^2}{1!} = t^2 \checkmark$

Show by induction that if $f(t,y) = 2t(1+y)$,

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds, \text{ and } \phi_0(t) = 0,$$

then $\phi_n(t) = \sum_{k=1}^{n} \frac{t^{2k}}{k!}$ for $n \geq 1$

Proof by induction on n:

For $n = 1$,

LHS

$$\phi_1(t) = \int_0^t f(s, \phi_0(s))ds = \int_0^t f(s, 0)ds$$
$$= \int_0^t 2s(1+0)ds = s^2|_0^t = t^2$$

RHS

$$\sum_{k=1}^{1} \frac{t^{2k}}{k!} = \frac{t^{2(1)}}{(1)!} = t^2 = \underline{\phi_1(t)}$$

LHS

Show by induction that if $f(t,y) = 2t(1+y)$,

$\phi_0(t) = 0$, and $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds$,

then $\phi_n(t) = \displaystyle\sum_{k=1}^{n} \frac{t^{2k}}{k!}$ for $n \geq 1$

Proof by induction on n.

$n = 1$: Claim: $\phi_1(t) = \displaystyle\sum_{k=1}^{1} \frac{t^{2k}}{k!}$

Alternative way of writing same thing

LHS

$$\phi_1(t) = \int_0^t f(s, \phi_0(s))ds = \int_0^t f(s, 0)ds$$

$$= \int_0^t 2s(1+0)ds = s^2|_0^t = t^2 = \frac{t^{2(1)}}{(1)!} = \sum_{k=1}^{1} \frac{t^{2k}}{k!}$$

RHS

Show by induction that if $f(t,y) = 2t(1+y)$,

$$\phi_0(t) = 0, \text{ and } \phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds,$$

then $\phi_n(t) = \sum_{k=1}^{n} \frac{t^{2k}}{k!}$ for $n \geq 1$ $\Leftarrow$ conclusion

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Induction hypothesis:

For $n = j-1$

$$\phi_{j-1}(t) = \sum_{k=1}^{j-1} \frac{t^{2k}}{k!}$$

Show by induction that if $f(t, y) = 2t(1 + y)$,

$$\phi_0(t) = 0, \text{ and } \phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds,$$

then $\phi_n(t) = \sum_{k=1}^{n} \frac{t^{2k}}{k!}$ for $n \geq 1$

Induction hypothesis:

Suppose for $n = j - 1$, $\quad \phi_{j-1}(t) = \sum_{k=1}^{j-1} \frac{t^{2k}}{k!}$

Show by induction that if $f(t,y) = 2t(1+y)$,

$\phi_0(t) = 0$, and $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds$,

then $\phi_n(t) = \sum_{k=1}^{n} \frac{t^{2k}}{k!}$ for $n \geq 1$

Induction hypothesis:

get an extra hypothesis to work with

Suppose for $n = j-1$, $\phi_{j-1}(t) = \sum_{k=1}^{j-1} \frac{t^{2k}}{k!}$

FYI

Hypothesis: $f(t,y) = 2t(1+y)$, $\phi_0(t) = 0$,

$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds$, and $\phi_{j-1}(t) = \sum_{k=1}^{j-1} \frac{t^{2k}}{k!}$

Show by induction that if $f(t, y) = 2t(1 + y)$,

$$\phi_0(t) = 0, \text{ and } \phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds,$$

then $\phi_n(t) = \sum_{k=1}^{n} \frac{t^{2k}}{k!}$ for $n \geq 1$

Induction hypothesis:

Suppose for $n = j - 1$, $\quad \phi_{j-1}(t) = \sum_{k=1}^{j-1} \frac{t^{2k}}{k!}$

$j-1 \Rightarrow j$

Claim: $\phi_j = \sum_{k=1}^{j} \frac{t^{2k}}{k!}$ $\leftarrow S(j)$ $\leftarrow$ GOAL

3 Hypothesis: $f(t,y) = 2t(1+y)$, $\phi_0(t) = 0$,

induct hyp

1 $\boxed{\phi_{n+1}(t) = \int_0^t f(s,\phi_n(s))\,ds}$, and $\phi_{j-1}(t) = \sum_{k=1}^{j-1} \frac{t^{2k}}{k!}$ 2

Claim: $\phi_j = \sum_{k=1}^{j} \frac{t^{2k}}{k!}$

pf of Claim: $\phi_j(t) = \int_0^t f(s, \phi_{j-1}(s))\,ds$

$= \int_0^t f\left(s, \sum_{k=1}^{j-1} \frac{s^{2k}}{k!}\right) ds = \int_0^t 2s\left(1+\sum_{k=1}^{j-1} \frac{s^{2k}}{k!}\right) ds$

Hypothesis: $f(t,y) = 2t(1+y)$; $\phi_0(t) = 0$,

2

1 $\phi_{n+1}(t) = \int_0^t f(s,\phi_n(s))ds$, and $\phi_{j-1}(t) = \displaystyle\sum_{k=1}^{j-1} \frac{t^{2k}}{k!}$ 3

Claim: $\phi_j = \displaystyle\sum_{k=1}^{j} \frac{t^{2k}}{k!}$

Proof of claim: $\phi_j = \int_0^t f(s,\phi_{j-1}(s))ds$

$$= \int_0^t 2s(1+\phi_{j-1}(s))ds$$

$$= \int_0^t 2s\left(1+\sum_{k=1}^{j-1} \frac{s^{2k}}{k!}\right)ds$$

Claim: $\phi_j = \sum_{k=1}^{j} \frac{t^{2k}}{k!}$

Proof of claim: $\phi_j = \int_0^t f(s, \phi_{j-1}(s))ds$

$$= \int_0^t 2s(1 + \phi_{j-1}(s))ds$$

$$= \int_0^t 2s(1 + \sum_{k=1}^{j-1} \frac{s^{2k}}{k!})ds$$

Claim: $\phi_j = \sum_{k=1}^{j} \frac{t^{2k}}{k!}$

Proof of claim: $\phi_j = \int_0^t f(s, \phi_{j-1}(s))ds$

$$= \int_0^t 2s(1 + \phi_{j-1}(s))ds$$

$$= \int_0^t 2s(1 + \sum_{k=1}^{j-1} \frac{s^{2k}}{k!})ds$$

$$= \int_0^t (2s + \sum_{k=1}^{j-1} \frac{2s^{2k+1}}{k!})ds$$

$$= \int_0^t (\sum_{k=0}^{j-1} \frac{2s^{2k+1}}{k!}) ds$$

$$\boxed{K = 0 \quad \frac{2s^1}{0!} = 2s}$$

Claim: $\phi_j = \sum_{k=1}^{j} \frac{t^{2k}}{k!}$

Proof of claim: $\phi_j = \int_0^t f(s, \phi_{j-1}(s))ds$

$$= \int_0^t 2s(1 + \phi_{j-1}(s))ds$$

$$= \int_0^t 2s(1 + \sum_{k=1}^{j-1} \frac{s^{2k}}{k!})ds$$

$$= \int_0^t (2s + \sum_{k=1}^{j-1} \frac{2s^{2k+1}}{k!})ds$$

$$= \int_0^t (\sum_{k=0}^{j-1} \frac{2s^{2k+1}}{k!})ds$$

Alternative (c!) could have integrated first

combined into sing $\sum_{k=0}^{j-1}$

Claim: $\phi_j = \sum_{k=1}^{j} \frac{t^{2k}}{k!}$

$$\int_0^t \left(\sum_{k=0}^{j-1} \frac{2s^{2k+1}}{k!}\right) ds = \sum_{k=0}^{j-1} \frac{2t^{2k+2}}{(2k+2)k!}$$

integrated

plugged in t & 0

simplify

$$= \sum_{k=0}^{j-1} \frac{\cancel{2}t^{2k+2}}{\cancel{2}(k+1)k!}$$

simplify

$$= \sum_{k=0}^{j-1} \frac{t^{2k+2}}{(k+1)!} = \sum_{k=1}^{j} \frac{t^{2k}}{k!}$$

$$= \sum_{k=0+1}^{j-1+1} \frac{t^{2(k-1)+1}}{(k-1+1)!}$$

//

Claim: $\phi_j = \sum_{k=1}^{j} \frac{t^{2k}}{k!}$

$$\int_0^t \Big(\sum_{k=0}^{j-1} \frac{2s^{2k+1}}{k!}\Big) ds = \sum_{k=0}^{j-1} \frac{2t^{2k+2}}{(2k+2)k!}$$

$$= \sum_{k=0}^{j-1} \frac{2t^{2k+2}}{2(k+1)k!}$$

$$= \sum_{k=0}^{j-1} \frac{t^{2k+2}}{(k+1)!} = \sum_{k=1}^{j} \frac{t^{2k}}{k!}$$

goal

$\square$

Show by induction that if $f(t,y) = 2t(1+y)$,

$$\phi_0(t) = 0, \text{ and } \phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds,$$

then $\phi_n(t) = \sum_{k=1}^{n} \frac{t^{2k}}{k!}$ for $n \geq 1$

LHS RHS

Proof by induction on n.

$n = 1$: Claim: $\phi_1(t) = \sum_{k=1}^{1} \frac{t^{2k}}{k!}$

base case

LHS

$$\phi_1(t) = \int_0^t f(s, \phi_0(s))ds = \int_0^t f(s, 0)ds$$

$$= \int_0^t 2s(1+0)ds = s^2\big|_0^t = t^2 = \frac{t^{2(1)}}{(1)!} = \sum_{k=1}^{1} \frac{t^{2k}}{k!}$$

RHS

Show by induction that if $f(t,y) = 2t(1+y)$,

$\phi_0(t) = 0$, and $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds$,

then $\phi_n(t) = \sum_{k=1}^{n} \frac{t^{2k}}{k!}$ for $n \geq 1$

Induction hypothesis:

Suppose for $n = j-1$, $\phi_{j-1}(t) = \sum_{k=1}^{j-1} \frac{t^{2k}}{k!}$

Claim: $\phi_j = \sum_{k=1}^{j} \frac{t^{2k}}{k!}$

$S(j-1) \Rightarrow S(j))$

Claim: $\phi_j = \sum_{k=1}^{j} \frac{t^{2k}}{k!}$

Proof of claim: $\phi_j = \int_0^t f(s, \phi_{j-1}(s))ds$

LHS

$$= \int_0^t 2s(1 + \phi_{j-1}(s))ds$$

$$= \int_0^t 2s(1 + \sum_{k=1}^{j-1} \frac{s^{2k}}{k!})ds$$

$$= \int_0^t (2s + \sum_{k=1}^{j-1} \frac{2s^{2k+1}}{k!})ds$$

$$= \int_0^t (\sum_{k=0}^{j-1} \frac{2s^{2k+1}}{k!})ds$$

$$= \sum_{k=0}^{j-1} \frac{2t^{2k+2}}{(2k+2)k!}$$

$$= \sum_{k=0}^{j-1} \frac{2t^{2k+2}}{2(k+1)k!}$$

$$= \sum_{k=0}^{j-1} \frac{t^{2k+2}}{(k+1)!}$$

$$= \sum_{k=1}^{j} \frac{t^{2k}}{k!}$$

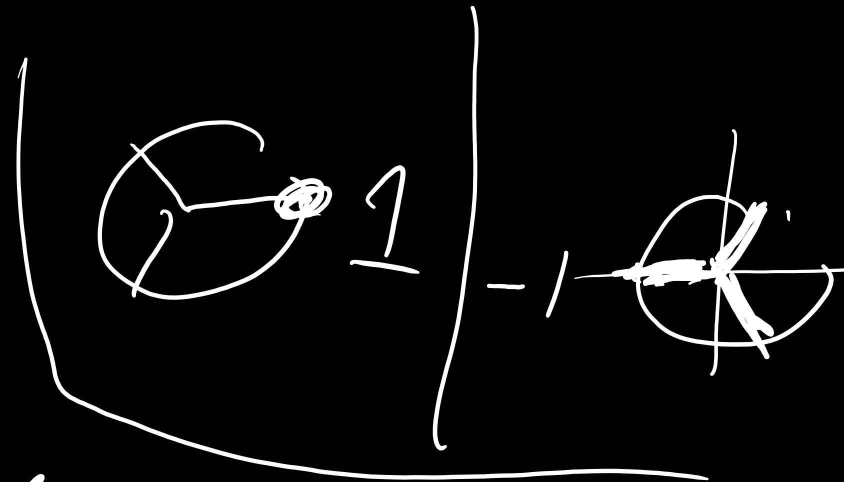
RHS

Ch 3 & 4

Step 1: Solve homog

factor polynomial

- standard factoring
- quadratic formula
- long division (can do by inspection)
- roots of unity (pictorially & algebraically)



Wronskian
l.i
fundamental set of soln
basis

Abel's thm

Step 2: Find a nonhomog soln

General soln $y = c_1\phi_1 + \ldots + c_n\phi_n + \psi$

Step 3: If IVP find c_i by plugging in initial values