nth order LINEAR differential equation:

Thm 2.4.1: If p and g are continuous on (a, b) and the point $t_0 \in (a, b)$, then there exists a unique function $y = \phi(t)$ defined on (a, b) that satisfies the following initial value problem:

$$y' + p(t)y = g(t), \quad y(t_0) = y_0.$$

Thm 3.2.1: If $p : (a, b) \to R$, $q : (a, b) \to R$, and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$y'' + p(t)y' + q(t)y = g(t),$$
$$y(t_0) = y_0, \quad y'(t_0) = y_1$$

hypothesis

open interval

Theorem 4.1.1: If $p_i : (a, b) \to R$, $i = 1, ..., n$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$y^{(n)} + p_1(t) y^{(n-1)} + ... + p_{n-1}(t) y' + p_n(t) y = g(t),$$

$$y(t_0) = y_0, \quad y'(t_0) = y_1, \; ..., \quad y^{(n-1)}(t_0) = y_{n-1}$$

needed for theory

Proof: We proved the case $n = 1$ using an integrating factor. When $n > 1$, see more advanced textbook.

NOTE: Theorem 4.1.1 is VERY useful in the real world. Suppose you can't solve the linear differential equation directly. You may be able to instead approximate the solution – see for example ch 5 series solution (guess $y = \sum a_n x^n$), which we won't cover in this class or $\boxed{\text{MATH:3800}}$ Elementary Numerical Analysis.

or Picard's iteration

But your approximation is not of much use unless you know where your approximation is valid.

Determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist and be unique.

$$(1-t)(1+t^2)y''' + \ln|t-5|\,y' + 2y = \sqrt{t+4} \qquad y(0) = 3$$

$(1-t)(1+t^2)$ $\quad (1-t)(1+t^2)$ $\quad (1-t)(1+t^2)$ $\quad (1-t)(1+t^2)$

$p_1(t) = \dfrac{\ln|t-5|}{(1-t)(1+t^2)}$ is continuous

for all real # except $t=5$, $t=1$

$1+t^2 \neq 0$ $\forall$ real t

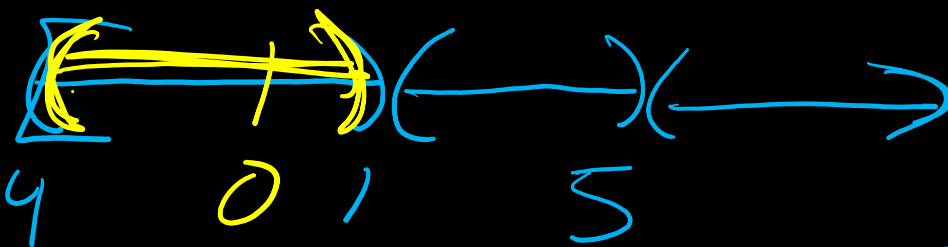
Domain of ~~soln~~ is subset of $\mathbb{R}$

Determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist and be unique.

$t_0 = 0$

$$(1-t)(1+t^2)y''' + \ln|t-5|\,y' + 2y = \sqrt{t+4} \qquad y(0) = 3$$

p_1, p_2, g are cont



$[-4, 1) \cup (1, 5) \cup (5, \infty)$

open interval containing $t_0 = 0 \Rightarrow \boxed{(-4, 1)}$

$t = 3 \Rightarrow (1, 5)$ | $t = 6 \Rightarrow (5, \infty)$

$t = -4, 5, 1, -6 \Rightarrow$ Thm does not apply

No info

Determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist and be unique.

$$(1-t)(1+t^2)y''' + \ln|t-5|\,y' + 2y = \sqrt{t+4} \qquad y(0) = 3$$

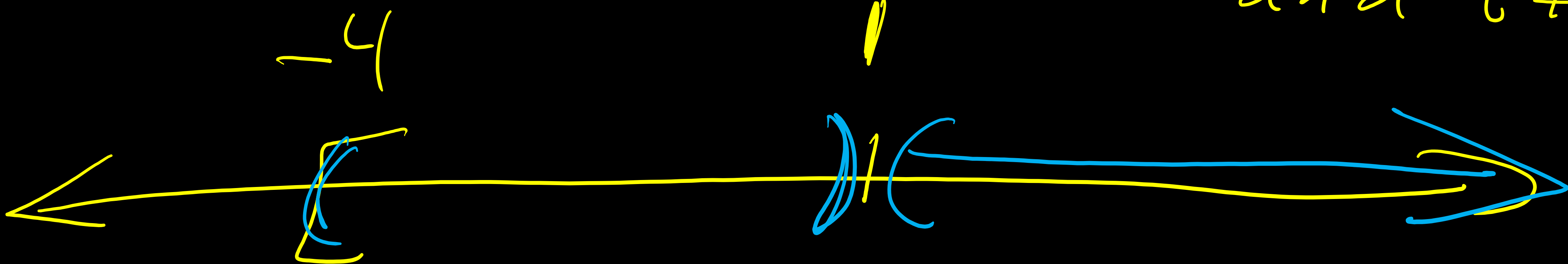
$$p_2(t) = \frac{2}{(1-t)(1+t^2)}$$ is cont $\forall\, t \neq 1$

Determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist and be unique.

$$(1-t)(1+t^2)y''' + \ln|t-5|\,y' + 2y = \sqrt{t+4} \qquad y(0) = 3$$

$$g(t) = \frac{\sqrt{t+4}}{(1-t)(1+t^2)}$$ is cont for $t \geq -4$ and $t \neq 1$

$[-4, 1) \cup (1, \infty)$



4.1: General Theory of nth Order Linear Eqns

When does the following IVP have a unique soln:

IVP: $y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = g(t)$,

$$y(t_0) = y_0,\ y'(t_0) = y_1, ..., y^{(n-1)}(t_0) = y_{n-1}.$$

Suppose $y = c_1\phi_1(t) + c_2\phi_2(t) + ... + c_n\phi_n(t) + \psi(t)$ is the general solution to DE. Then

see thm 4.1.1

or look at wronskian

but lets look at general soln and compare to wronskian

$$y(t_0) = y_0:$$
$$y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0) + \dots + c_n\phi_n(t_0) + \psi(t_0)$$

Thm 4.1.1 says
if fns are ~~cont~~
$\Rightarrow$ unique soln

$$y'(t_0) = y_1:$$
$$y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0) + \dots + c_n\phi_n'(t_0) + \psi'(t_0)$$
$$\vdots$$

n eqns from
n initial ~~valu~~

n unknowns
$c_1, \dots c_n$

$$y^{(n-1)}(t_0) = y_{n-1}:$$
$$y_{n-1} = c_1\phi_1^{(n-1)}(t_0) + c_2\phi_2^{(n-1)}(t_0) + \dots + c_n\phi_n^{(n-1)}(t_0) + \psi^{(n-1)}(t_0)$$

plug in initial value

Let $b_k = y_k - \psi^{(k)}(t_0)$. Note that in these equations the c_i are the unknowns

Translating this linear system of eqns into matrix form:

$$\underbrace{\begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & \dots & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & \dots & \phi_n'(t_0) \\ & \vdots & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & \dots & \phi_n^{(n-1)}(t_0) \end{bmatrix}}_{W(\phi_1, \dots \phi_n)(t_0)} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}$$

Note this equation has a unique solution if and only if

$$det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & \dots & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & \dots & \phi_n'(t_0) \\ & \vdots & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & \dots & \phi_n^{(n-1)}(t_0) \end{bmatrix} \neq 0$$

Defn: The Wronskian of the functions, ϕ_1, ϕ_2,..., ϕ_n is

$$W(\phi_1, \phi_2, ..., \phi_n)(t) = det \begin{bmatrix} \phi_1(t) & \phi_2(t) & \dots & \phi_n(t) \\ \phi_1'(t) & \phi_2'(t) & \dots & \phi_n'(t) \\ & & \vdots & \\ \phi_1^{(n-1)}(t) & \phi_2^{(n-1)}(t) & \dots & \phi_n^{(n-1)}(t) \end{bmatrix}$$

$$\boxed{\text{Note: } \{\phi_1, \phi_2, ..., \phi_n\} \text{ is a linearly independent set of fns if } W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0 \text{ for some } t_0}$$

det of coef matrix evaluat
at an arbitrary t

In other words if ϕ_i are homogeneous solutions to an nth order linear DE,

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0$$

and $W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0$ for some t_0.

iff $\{\phi_1, \phi_2, ..., \phi_n\}$ is a basis for the solution set of this homogeneous equation.

homog gen $y = c_1\phi_1 + ... + c_n\phi_n$

In other words any homogeneous solution can be written as a linear combination of these basis elements:

$$\boxed{y = c_1\phi_1 + ... + c_n\phi_n}$$

Moreover, the general soln to the non-homogeneous eqn

$$y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_{n-1}(t)y' + p_n(t)y = g(t)$$

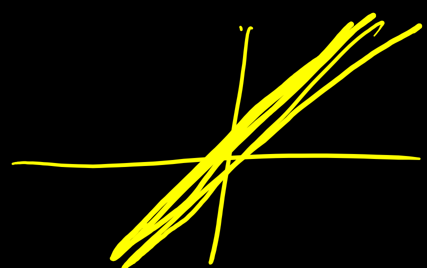
is just the translated version of the general homogeneous solution:

$$\boxed{y = c_1\phi_1 + \dots + c_n\phi_n + \psi}$$

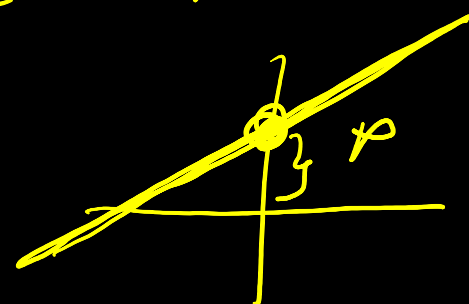
where ψ is a non-homogeneous solution.

(compare to MATH 2700

$Ax = 0$



$Ax = b$



In other words if ϕ_i are homogeneous solutions to an nth order linear DE,

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0$$

and $W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0$ for some t_0.

iff $\{\phi_1, \phi_2, ..., \phi_n\}$ is a basis for the solution set of this homogeneous equation.

In other words any homogeneous solution can be written as a linear combination of these basis elements:

Step 1 $$y = c_1\phi_1 + ... + c_n\phi_n$$

Moreover, the general soln to the non-homogeneous eqn

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = g(t)$$

is just the translated version of the general homogeneous solution:

Step 2: find ψ $$y = c_1\phi_1 + ... + c_n\phi_n + \psi$$

where ψ is a non-homogeneous solution.

Linear Independence and the Wronskian

$\phi_1, \ldots, \phi_n$ are linearly independent

iff ← defn $c_i = 0 \; \forall i$

$c_1\phi_1(t) + \ldots + c_n\phi_n(t) = 0$ has a unique solution (that works for all t).

Defn from MATH 2700

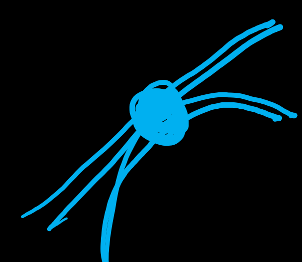
iff

the following system of equations has a unique solution

$$
\begin{aligned}
c_1\phi_1(t) + c_2\phi_2(t) + \ldots + c_n\phi_n(t) &= 0 \\
c_1\phi_1'(t) + c_2\phi_2'(t) + \ldots + c_n\phi_n'(t) &= 0 \\
&\vdots \\
c_1\phi_1^{(n-1)}(t) + c_2\phi_2^{(n-1)}(t) + \ldots + c_n\phi_n^{(n-1)}(t) &= 0
\end{aligned}
$$

take derivatives

This is why our initial values are
$y(t_0) = y_0$
$y'(t_0) = y_1$
$\vdots$
$y^{(n-1)} = y_{n-1}$



iff the following system of equations has a unique solution

$$\begin{bmatrix} \phi_1(t) & \phi_2(t) & \dots & \phi_n(t) \\ \phi_1'(t) & \phi_2'(t) & \dots & \phi_n'(t) \\ & \vdots & & \\ \phi_1^{(n-1)}(t) & \phi_2^{(n-1)}(t) & \dots & \phi_n^{(n-1)}(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Note this equation has a unique solution if and only if for some t_0

$$det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & \dots & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & \dots & \phi_n'(t_0) \\ & \vdots & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & \dots & \phi_n^{(n-1)}(t_0) \end{bmatrix} \neq 0$$

iff $\boxed{W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0,}$

$\phi_1, ..., \phi_n$ are linearly independent

iff $\boxed{W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0,}$ for some t_0

Example: Determine if $\{1+2t, 5+4t^2, 6-8t+8t^2\}$ are linearly independent:

Method 1: MATH 2700 0 $\{1, t, t^2\}$

Solve $c_1(1+2t)+c_2(5+4t^2)+c_3(6-8t+8t^2)=0$

Or equivalently,

vector format

$$\text{solve } c_1\begin{bmatrix}1\\2\\0\end{bmatrix}+c_2\begin{bmatrix}5\\0\\4\end{bmatrix}+c_3\begin{bmatrix}6\\-8\\8\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Or equivalently, solve $\begin{bmatrix}1&5&6\\2&0&-8\\0&4&8\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

matrix format

Example: Determine if $\{1+2t, 5+4t^2, 6-8t+8t^2\}$ are linearly independent:

Method 2: Check the Wronskian

$$\underbrace{det \begin{bmatrix} 1+2t & 5+4t^2 & 6-8t+8t^2 \\ 2 & 8t & -8+16t \\ 0 & 8 & 16 \end{bmatrix}}$$

This works even if fns are not polynomials (or linear comb of linearly indep fns)

Method 2: Check the Wronskian

$$det \begin{bmatrix} 1+2t & 5+4t^2 & 6-8t+8t^2 \\ 2 & 8t & -8+16t \\ 0 & 8 & 16 \end{bmatrix}$$

$$+(1+2t)\begin{vmatrix} 8t & -8+16t \\ 8 & 16 \end{vmatrix} - 2 \begin{vmatrix} 5+4t^2 & 6-8t+8t^2 \\ 8 & 16 \end{vmatrix} + 0$$

$$= (1+2t)\left(8 \cdot 16t - 8(-8+16t)\right) - 2\left[16(5+4t^2) - 8(6-8t+8t^2)\right]$$

$$= \text{etc} \ldots \neq 0 \text{ for some } t_0$$

$$\Rightarrow \text{l.i.}$$

Abel's theorem: if ϕ_i are homogeneous solutions to an nth order linear DE,

$$1\, y^{(n)} + p_1(t)y^{(n-1)} \cancel{+ \dots + p_{n-1}(t)y' + p_n(t)y = 0}$$

then $\underline{W(\phi_1, \phi_2, ..., \phi_n)(t)} = \boxed{ce^{-\int p_1(t)dt}}$ for some constant c

n th order lin DE

coef of $y^{(n-1)}$

Ex: Find the Wronskian of a fundamental set of solutions of the DE

old method

$$\boxed{y'' + 5y' = 0}$$

$$W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}$$

Method 1: Find homogeneous solution

$r^2 + 5r = 0$ implies $r = 0, -5$

homog sol'n $\boxed{y = c_1e^{0t} + c_2e^{-5t}} = c_1(1) + c_2(e^{-5t}) = c_1 + c_2e^{-5t}$

A fundamental set of solutions: $\{1, e^{-5t}\}$

a basis for sol'n space

Wronskian $= W(1, e^{-5t})(t) = det \begin{pmatrix} 1 & e^{-5t} \\ 0 & -5e^{-5t} \end{pmatrix} = -5e^{-5t}$

Export

Method 2: Abel's theorem: Wronskian $= ce^{-\int p_1(t)dt}$

$1y'' + 5y' = 0$ implies $p_1(t) = 5$

$$W(\phi_1, \phi_2) = ce^{-\int 5\,dt} = \underline{ce^{-5t}}$$

$$y'' + 5y' + 6y = 0 \Rightarrow W(f_1, f_2)(t)$$

$$y'' + 5y' + \cos t\, y = \ln|t| \nearrow \quad = ce^{-5t}$$

$$\frac{2}{2} y^{iv} + \frac{10}{2} y''' + \frac{\sim\sim\sim}{2} = \sim\sim e^{-5}$$

$$\boxed{p_1(t) = 5} \qquad W(f_1, f_2, f_3, f_4)(t_0) = ce$$

Method 2: Abel's theorem: Wronskian $= ce^{-\int p_1(t)dt}$

$y'' + 5y' = 0$ implies $p_1(t) = 5$.

Thus Wronskian $= W(1, e^{-5t})(t) = ce^{-\int 5dt} = ce^{-5t}$

If we are given

$W(1, e^{-5t})(0) = -5 \Leftarrow$ plug in to find c

$-5 = ce^{-5(0)} \Rightarrow c = -5 \Rightarrow W = -5e^{-5t}$