Solving polynomial equations:

$$\text{Example: } r^3 + r^2 + 3r + 10 = 0$$

Plug in $r = \pm 1, \pm 2, \pm 5, \pm 10$ to see if any of these are solns:

$$(\pm 1)^3 + (\pm 1)^2 + 3(\pm 1) + 10 \neq 0$$

$$(\pm 2)^3 + (\pm 2)^2 + 3(\pm 2) + 10 \;?=?0$$

$$(-2)^3 + (-2)^2 + 3(-2) + 10 = -8 + 4 - 6 + 10 = 0$$

Thus $(r - (-2))$ is a factor of $r^3 + r^2 + 3r + 10$

Hence $r^3 + r^2 + 3r + 10 = (r + 2)(r^2 + \_\_\_\_\, r + 5)$

To find the coefficient of r in the above, you can do so by (1) long division, (2) inspection, (3) using variable x

$$r^3 + r^2 + 3r + 10 = (r+2)(r^2 + \underline{\quad x \quad}\, r + 5)$$

$$\begin{array}{rcccccccc} (r+2)(r^2 + \underline{\quad x \quad}\, r + 5) = & r^3 & + & (2+x)r^2 & + & (2x+5)r & + & 10 \\ & r^3 & + & r^2 & + & 3r & + & 10 \end{array}$$

Thus $2 + x = 1$ and hence $x = -1$

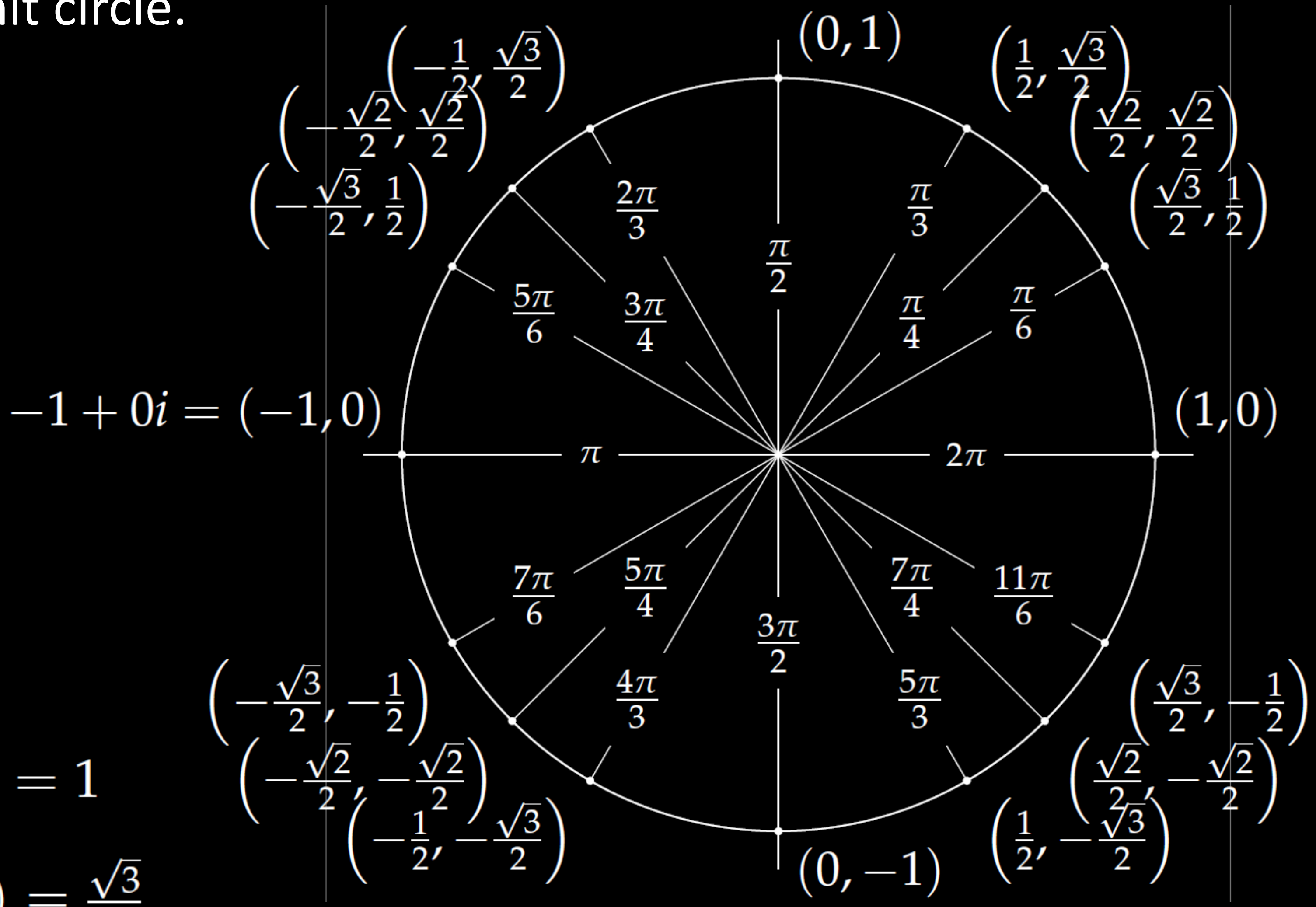
Check: $2x + 5 = 2(-1) + 5 = 3$

Hence $r^3 + r^2 + 3r + 10 = (r+2)(r^2 - r + 5) = 0$

Thus $r = -2, \frac{1 \pm \sqrt{1-20}}{2}$.

Thus $r = -2, \frac{1 \pm i\sqrt{19}}{2}$.

In special cases, you can use the unit circle.



$$sin(0)=\frac{\sqrt{0}}{2}=0$$

$$sin(\tfrac{\pi}{6})=\frac{\sqrt{1}}{2}=\frac{1}{2}$$

$$sin(\tfrac{\pi}{4})=\frac{\sqrt{2}}{2}$$

$$sin(\tfrac{\pi}{3})=\frac{\sqrt{3}}{2}$$

$$sin(\tfrac{\pi}{2})=\frac{\sqrt{4}}{2}=1$$

$$cos(0)=1$$

$$cos(\tfrac{\pi}{6})=\frac{\sqrt{3}}{2}$$

$$cos(\tfrac{\pi}{4})=\frac{\sqrt{2}}{2}$$

$$cos(\tfrac{\pi}{3})=\frac{1}{2}$$

$$cos(\tfrac{\pi}{2})=0$$

In special cases, you can use the unit circle.

Ex: $r^4 + 1 = 0$ implies

$$r = (-1)^{\frac{1}{4}} = (-1 + 0i)^{\frac{1}{4}} = (e^{i\pi})^{\frac{1}{4}} = (e^{i(\pi+2\pi k)})^{\frac{1}{4}}$$

$$k = 0 : \qquad e^{\frac{i\pi}{4}} = cos(\tfrac{\pi}{4}) + isin(\tfrac{\pi}{4}) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$k = 1 : \qquad e^{\frac{3i\pi}{4}} = cos(\tfrac{3\pi}{4}) + isin(\tfrac{3\pi}{4}) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$k = 2 : \qquad e^{\frac{5i\pi}{4}} = cos(\tfrac{5\pi}{4}) + isin(\tfrac{5\pi}{4}) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

$$k = 3 : \qquad e^{\frac{7i\pi}{4}} = cos(\tfrac{7\pi}{4}) + isin(\tfrac{7\pi}{4}) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

$$r = (-1)^{\frac{1}{4}} = (-1+0i)^{\frac{1}{4}} = (e^{i\pi})^{\frac{1}{4}} = (e^{i(\pi+2\pi k)})^{\frac{1}{4}}$$

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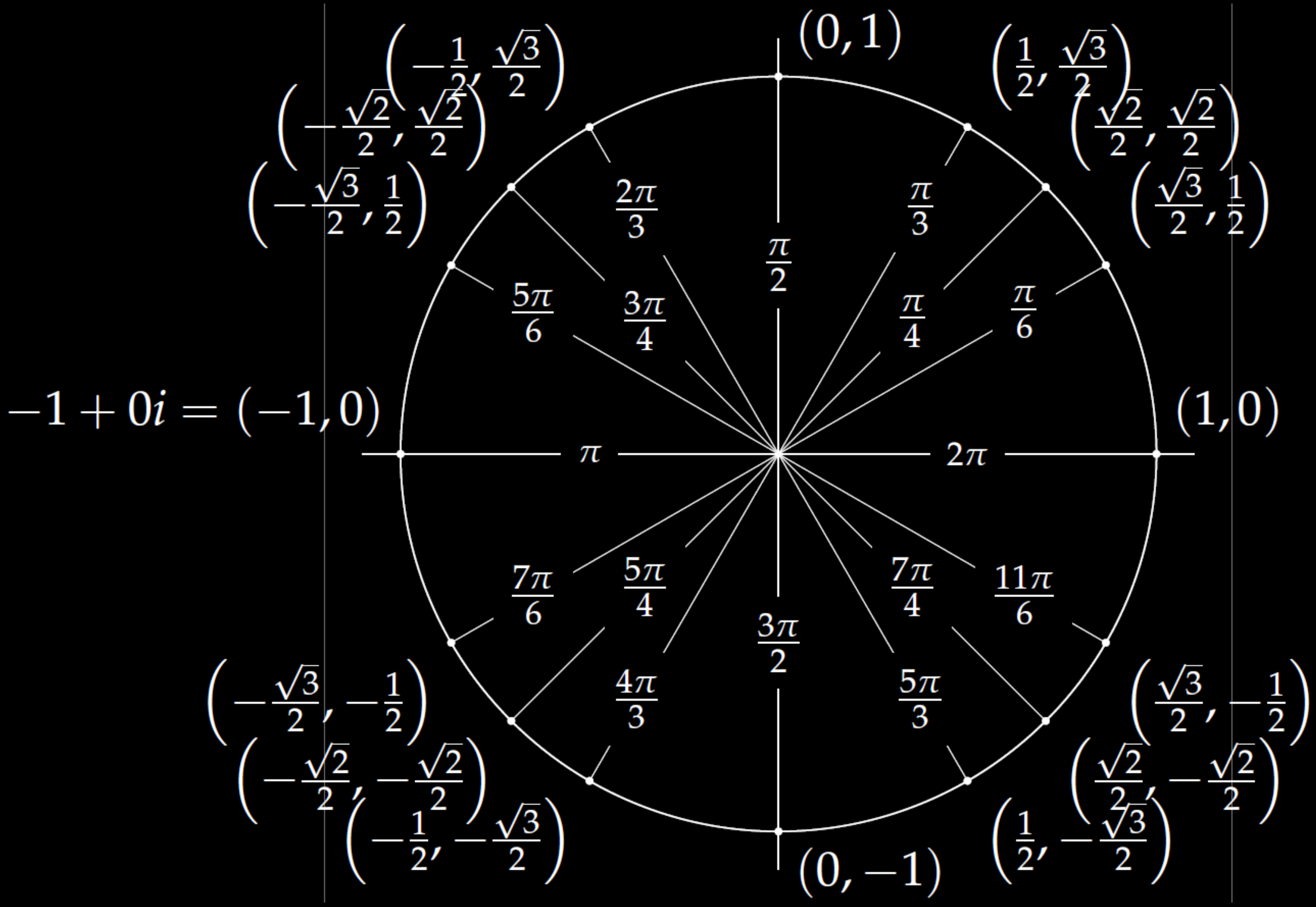
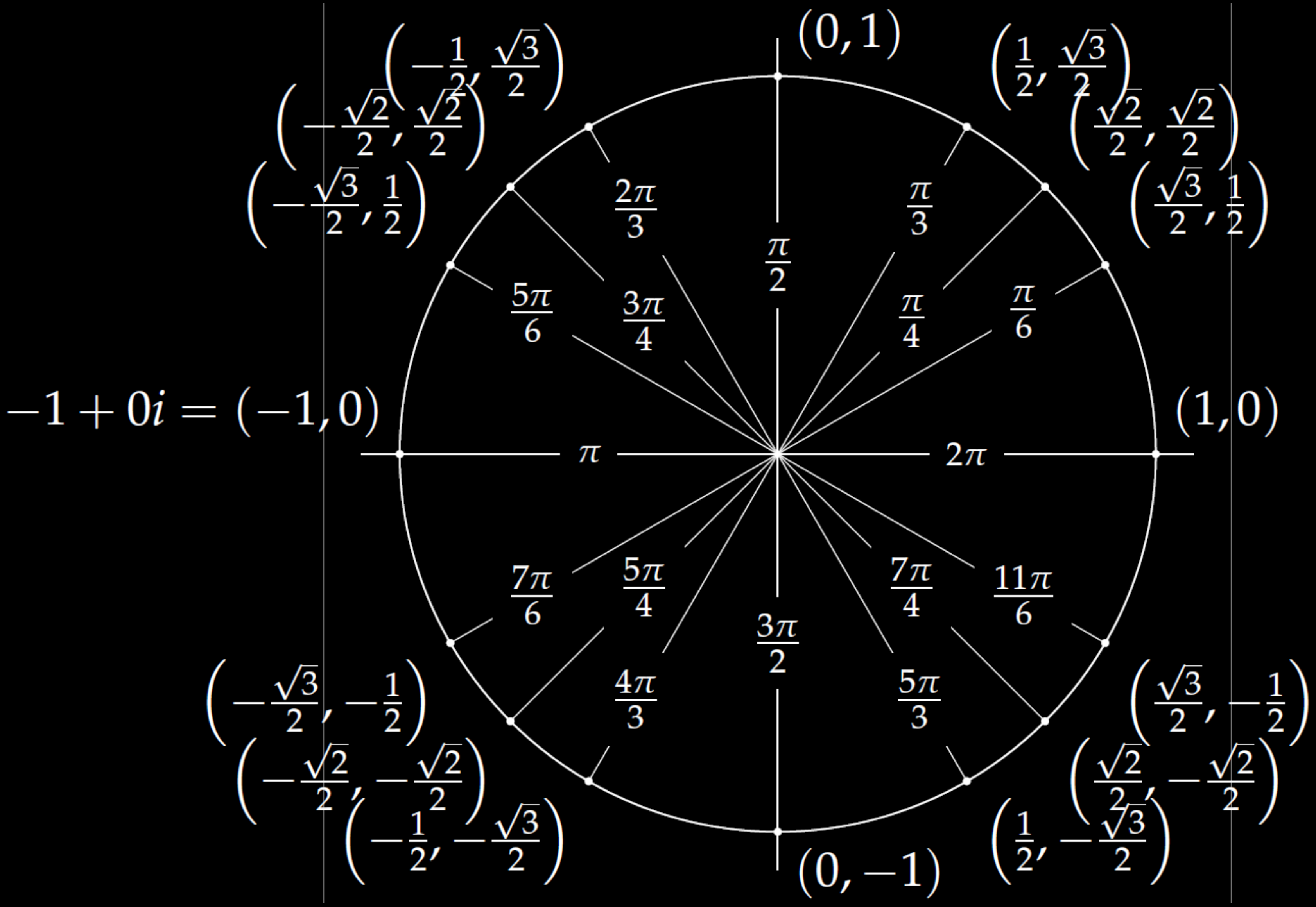


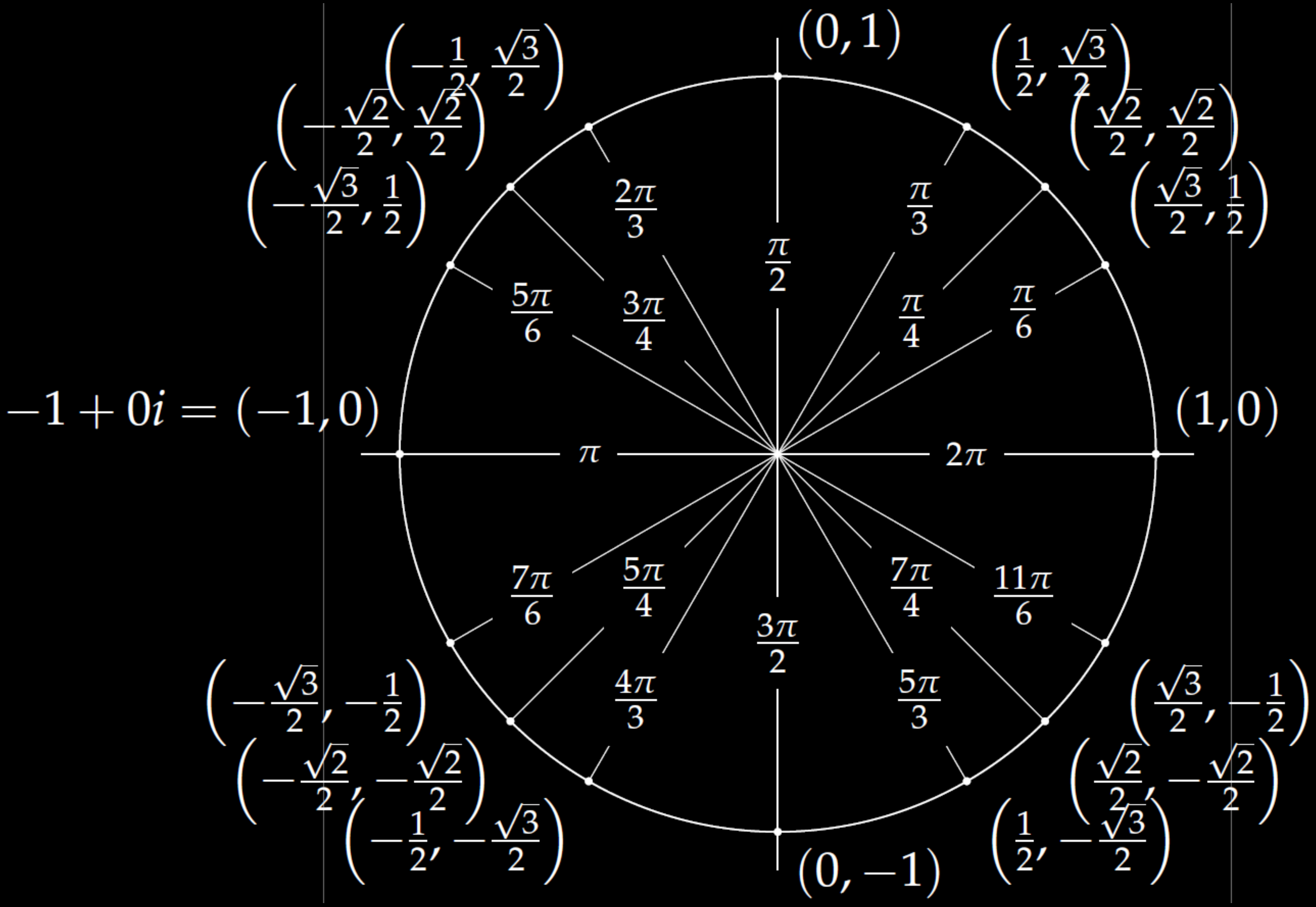
$$r = (-1)^{\frac{1}{4}} = (-1+0i)^{\frac{1}{4}} = (e^{i\pi})^{\frac{1}{4}} = (e^{i(\pi+2\pi k)})^{\frac{1}{4}}$$

$$k = 1: \qquad e^{\frac{3i\pi}{4}} = cos(\frac{3\pi}{4}) + isin(\frac{3\pi}{4}) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$



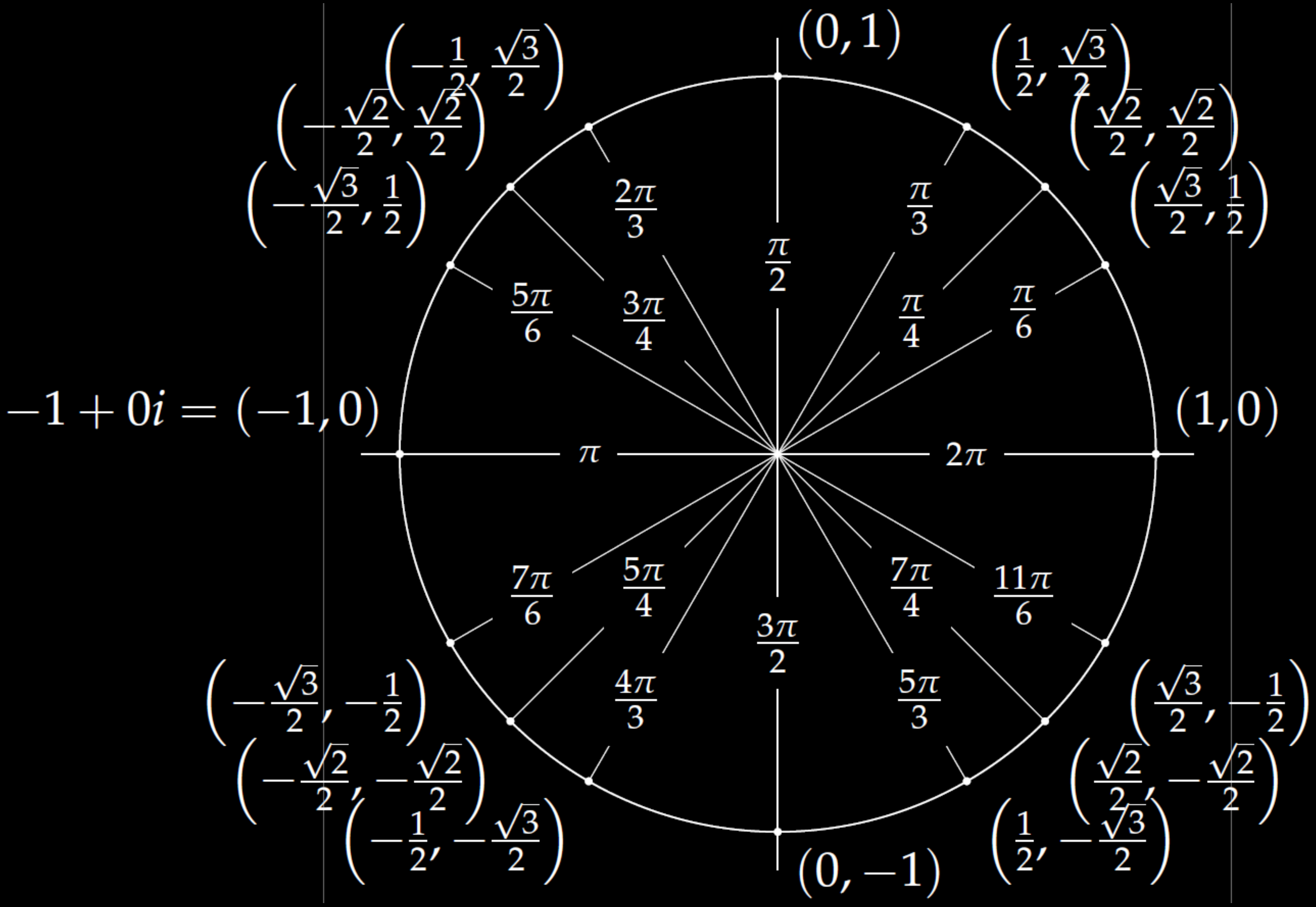
$$r = (-1)^{\frac{1}{4}} = (-1+0i)^{\frac{1}{4}} = (e^{i\pi})^{\frac{1}{4}} = (e^{i(\pi+2\pi k)})^{\frac{1}{4}}$$

$$k = 2: \qquad e^{\frac{5i\pi}{4}} = cos(\frac{5\pi}{4}) + isin(\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

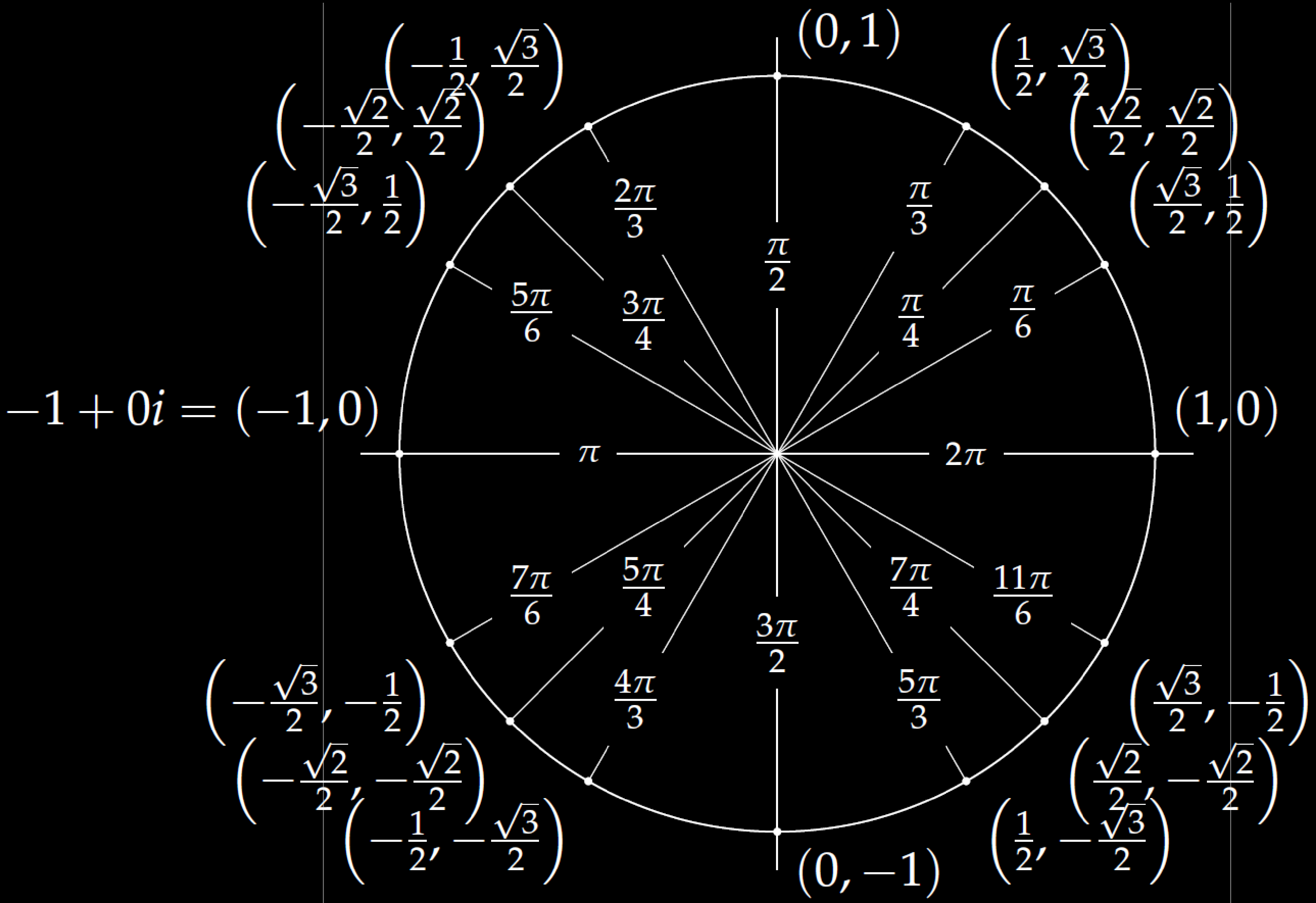


$$r = (-1)^{\frac{1}{4}} = (-1+0i)^{\frac{1}{4}} = (e^{i\pi})^{\frac{1}{4}} = (e^{i(\pi+2\pi k)})^{\frac{1}{4}}$$

$$k = 3: \qquad e^{\frac{7i\pi}{4}} = cos(\frac{7\pi}{4}) + isin(\frac{7\pi}{4}) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$



$$r = (-1)^{\frac{1}{4}} = (-1+0i)^{\frac{1}{4}} = \left(e^{i\pi}\right)^{\frac{1}{4}} = \left(e^{i(\pi+2\pi k)}\right)^{\frac{1}{4}}$$



In special cases, you can use the unit circle.

Ex: $r^4+1=0$ implies

$$r=(-1)^{\frac{1}{4}}=(-1+0i)^{\frac{1}{4}}=(e^{i\pi})^{\frac{1}{4}}=(e^{i(\pi+2\pi k)})^{\frac{1}{4}}$$

$$k=0: \qquad e^{\frac{i\pi}{4}}=cos(\tfrac{\pi}{4})+isin(\tfrac{\pi}{4})=\tfrac{\sqrt{2}}{2}+i\tfrac{\sqrt{2}}{2}$$

$$k=1: \qquad e^{\frac{3i\pi}{4}}=cos(\tfrac{3\pi}{4})+isin(\tfrac{3\pi}{4})=-\tfrac{\sqrt{2}}{2}+i\tfrac{\sqrt{2}}{2}$$

$$k=2: \qquad e^{\frac{5i\pi}{4}}=cos(\tfrac{5\pi}{4})+isin(\tfrac{5\pi}{4})=-\tfrac{\sqrt{2}}{2}-i\tfrac{\sqrt{2}}{2}$$

$$k=3: \qquad e^{\frac{7i\pi}{4}}=cos(\tfrac{7\pi}{4})+isin(\tfrac{7\pi}{4})=\tfrac{\sqrt{2}}{2}-i\tfrac{\sqrt{2}}{2}$$

Example: Solve $y^{(iv)} + y = 0$

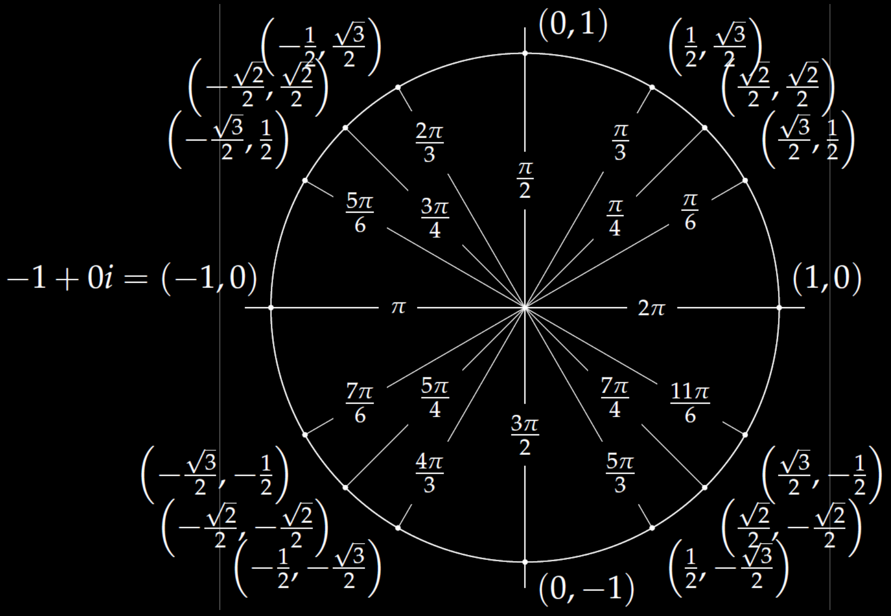
$y = e^{rt}$ implies $r^4 + 1 = 0$ and thus

$$r = \frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2} \text{ and } r = -\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}$$

Thus general homogeneous solution is

$$y = c_1 e^{\frac{\sqrt{2}}{2}t} cos(\frac{\sqrt{2}}{2}t) + c_2 e^{\frac{\sqrt{2}}{2}t} sin(\frac{\sqrt{2}}{2}t)$$

$$+ c_3 e^{-\frac{\sqrt{2}}{2}t} cos(\frac{\sqrt{2}}{2}t) + c_4 e^{-\frac{\sqrt{2}}{2}t} sin(\frac{\sqrt{2}}{2}t)$$



Example: Solve $y''' - 8y = 0$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$.

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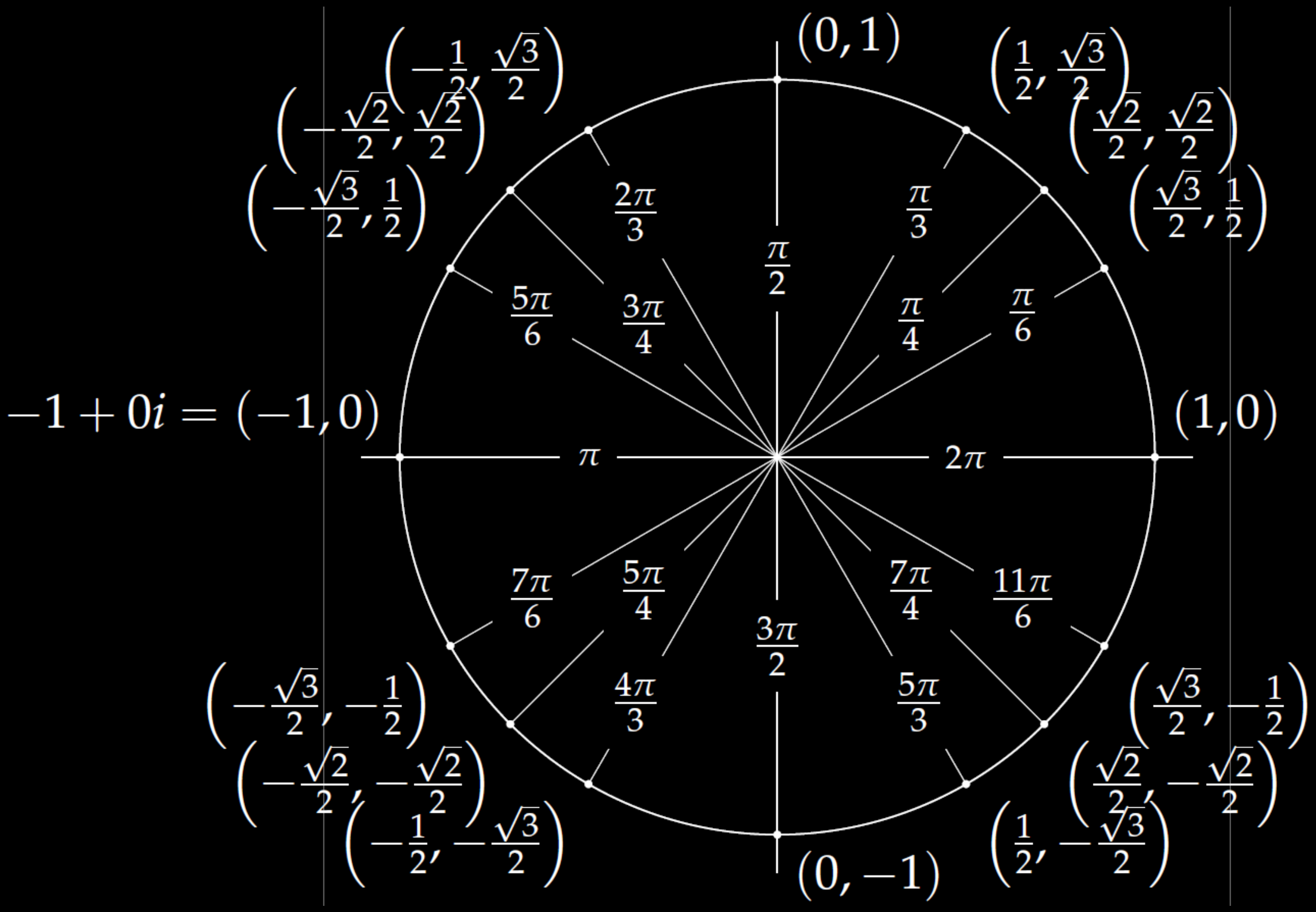

nth order LINEAR differential equation:

Thm 2.4.1: If p and g are continuous on (a, b) and the point $t_0 \in (a, b)$, then there exists a unique function $y = \phi(t)$ defined on (a, b) that satisfies the following initial value problem:

$$y' + p(t)y = g(t), \ \ y(t_0) = y_0.$$

Thm 3.2.1: If $p : (a, b) \to R$, $q : (a, b) \to R$, and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$y'' + p(t)y' + q(t)y = g(t),$$
$$y(t_0) = y_0, \ \ y'(t_0) = y_1$$

Theorem 4.1.1: If $p_i : (a,b) \to R$, $i = 1, ..., n$ and $g : (a,b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a,b) \to R$ that satisfies the initial value problem

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = g(t),$$
$$y(t_0) = y_0, \quad y'(t_0) = y_1, \ ..., \quad y^{(n-1)}(t_0) = y_{n-1}$$

Proof: We proved the case $n = 1$ using an integrating factor. When $n > 1$, see more advanced textbook.

NOTE: Theorem 4.1.1 is VERY useful in the real world. Suppose you can't solve the linear differential equation directly. You may be able to instead approximate the solution – see for example ch 5 series solution (guess $y = \sum a_n x^n$), which we won't cover in this class or MATH:3800 Elementary Numerical Analysis.

But your approximation is not of much use unless you know where your approximation is valid.

Determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist and be unique.

$$(1 - t)\left(1 + t^2\right)y''' + \ln|t - 5|\, y' + 2y = \sqrt{t+4} \qquad y(0) = 3$$

4.1: General Theory of nth Order Linear Eqns

When does the following IVP have a unique soln:

IVP: $y^{(n)}+p_1(t)y^{(n-1)}+...+p_{n-1}(t)y'+p_n(t)y = g(t)$,

$$y(t_0) = y_0,\ y'(t_0) = y_1, ..., y^{(n-1)}(t_0) = y_{n-1}.$$

Suppose $y = c_1\phi_1(t) + c_2\phi_2(t) + ... + c_n\phi_n(t) + \psi(t)$ is the general solution to DE. Then

$y(t_0) = y_0$:

$$y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0) + \ldots + c_n\phi_n(t_0) + \psi(t_0)$$

$y'(t_0) = y_1$:

$$y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0) + \ldots + c_n\phi_n'(t_0) + \psi'(t_0)$$

$$\vdots$$

$y^{(n-1)}(t_0) = y_{n-1}$:

$$y_{n-1} = c_1\phi_1^{(n-1)}(t_0) + c_2\phi_2^{(n-1)}(t_0) \\ + \ldots + c_n\phi_n^{(n-1)}(t_0) + \psi^{(n-1)}(t_0)$$

Let $b_k = y_k - \psi^{(k)}(t_0)$. Note that in these equations the c_i are the unknowns

Translating this linear system of eqns into matrix form:

$$\begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & \dots & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & \dots & \phi_n'(t_0) \\ & . & & \\ & . & & \\ & . & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & \dots & \phi_n^{(n-1)}(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ . \\ . \\ . \\ b_{n-1} \end{bmatrix}$$

Note this equation has a unique solution if and only if

$$det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & ... & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & ... & \phi_n'(t_0) \\ & \vdots & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & ... & \phi_n^{(n-1)}(t_0) \end{bmatrix} \neq 0$$

Defn: The Wronskian of the functions, ϕ_1, ϕ_2,..., ϕ_n is

$$W(\phi_1, \phi_2, ..., \phi_n) = det \begin{bmatrix} \phi_1(t) & \phi_2(t) & ... & \phi_n(t) \\ \phi_1'(t) & \phi_2'(t) & ... & \phi_n'(t) \\ & \cdot & & \\ & \cdot & & \\ & \cdot & & \\ \phi_1^{(n-1)}(t) & \phi_2^{(n-1)}(t) & ... & \phi_n^{(n-1)}(t) \end{bmatrix}$$

Note: $\{\phi_1, \phi_2, ..., \phi_n\}$ is a linearly independent set of fns
if $W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0$ for some t_0

In other words if ϕ_i are homogeneous solutions to an nth order linear DE,

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0$$

and $W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0$ for some t_0.

iff $\{\phi_1, \phi_2, ..., \phi_n\}$ is a basis for the solution set of this homogeneous equation.

In other words any homogeneous solution can be written as a linear combination of these basis elements:

$$y = c_1\phi_1 + ... + c_n\phi_n$$

Moreover, the general soln to the non-homogeneous eqn

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = g(t)$$

is just the translated version of the general homogeneous solution:

$$y = c_1\phi_1 + ... + c_n\phi_n + \psi$$

where ψ is a non-homogeneous solution.

In other words if ϕ_i are homogeneous solutions to an nth order linear DE,

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0$$

and $W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0$ for some t_0.

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is just the translated version of the general homogeneous solution:

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where ψ is a non-homogeneous solution.

Linear Independence and the Wronskian

$\phi_1, ..., \phi_n$ are linearly independent

iff

$c_1\phi_1(t) + ... + c_n\phi_n(t) = 0$ has a unique solution (that works for all t).

iff

the following system of equations has a unique solution

$$c_1\phi_1(t) + c_2\phi_2(t) + ... + c_n\phi_n(t) = 0$$

$$c_1\phi_1'(t) + c_2\phi_2'(t) + ... + c_n\phi_n'(t) = 0$$

$$\vdots$$

$$c_1\phi_1^{(n-1)}(t) + c_2\phi_2^{(n-1)}(t) + ... + c_n\phi_n^{(n-1)}(t) = 0$$

iff the following system of equations has a unique solution

$$\begin{bmatrix} \phi_1(t) & \phi_2(t) & \ldots & \phi_n(t) \\ \phi_1'(t) & \phi_2'(t) & \ldots & \phi_n'(t) \\ & . & & \\ & . & & \\ & . & & \\ \phi_1^{(n-1)}(t) & \phi_2^{(n-1)}(t) & \ldots & \phi_n^{(n-1)}(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

Note this equation has a unique solution if and only if for some t_0

$$det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & \dots & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & \dots & \phi_n'(t_0) \\ & . & & \\ & . & & \\ & . & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & \dots & \phi_n^{(n-1)}(t_0) \end{bmatrix} \neq 0$$

$$\text{iff} \quad W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0,$$

Example: Determine if $\{1+2t, 5+4t^2, 6-8t+8t^2\}$ are linearly independent:

Method 1:

Solve $c_1(1+2t) + c_2(5+4t^2) + c_3(6-8t+8t^2) = 0$

Or equivalently,

$$\text{solve } c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 6 \\ -8 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Or equivalently, solve $\begin{bmatrix} 1 & 5 & 6 \\ 2 & 0 & -8 \\ 0 & 4 & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Example: Determine if $\{1+2t, 5+4t^2, 6-8t+8t^2\}$ are linearly independent:

Method 2: Check the Wronskian

$$det \begin{bmatrix} 1+2t & 5+4t^2 & 6-8t+8t^2 \\ 2 & 8t & -8+16t \\ 0 & 8 & 16 \end{bmatrix}$$

Method 2: Check the Wronskian

$$det \begin{bmatrix} 1+2t & 5+4t^2 & 6-8t+8t^2 \\ 2 & 8t & -8+16t \\ 0 & 8 & 16 \end{bmatrix}$$

Abel's theorem: if ϕ_i are homogeneous solutions to an nth order linear DE,

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0$$

then $W(\phi_1, \phi_2, ..., \phi_n)(t) = ce^{-\int p_1(t)dt}$ for some constant c