So why did we guess $y = e^{rt}$?

Goal: Solve linear homogeneous 2nd order DE with constant coefficients,

$$ay'' + by' + cy = 0 \text{ where } a, b, c \text{ are constants}$$

Standard mathematical technique: make up simpler problems and see if you can generalize to the problem of interest.

Ex: linear homogeneous 1rst order DE: $y' + 2y = 0$

integrating factor $u(t) = e^{\int 2dt} = e^{2t}$

$y'e^{2t} + 2e^{2t}y = 0$

$(e^{2t}y)' = 0$. Thus $\int (e^{2t}y)'dt = \int 0dt$. Hence $e^{2t}y = C$

So $y = Ce^{-2t}$.

Thus exponential function could also be a solution to a linear homogeneous 2nd order DE

Ex: Simple linear homog 2nd order DE $y'' + 2y' = 0$.

Let $v = y'$, then $v' = y''$

$y'' + 2y' = 0$ implies $v' + 2v = 0$

Thus $v = y' = \frac{dy}{dt} = c_1e^{-2t}$.

$y'(t) = \frac{dy}{dt} = c_1 e^{-2t}$.

To find c_1, we need to know initial value $y'(t_0) = y_1$

Separate variables: $\quad dy = c_1 e^{-2t} dt$

$$y = c_1 e^{-2t} + c_2.$$

Note 2 integrations give us 2 constants, c_1 and c_2

To find constants, we need initial values, $y(t_0) = y_0$ and $y'(t_0) = y_1$

Note also that the general solution is a linear combination of two solutions:

Let $c_1 = 1,\ c_2 = 0$, then we see, $y(t) = e^{-2t}$ is a solution.

Let $c_1 = 0,\ c_2 = 1$, then we see, $y(t) = 1$ is a solution.

The general solution is a linear combination of two solutions:

$$y = c_1 e^{-2t} + c_2(1).$$

Recall: you have seen this before:

Solve linear homogeneous matrix equation $A\mathbf{y} = \mathbf{0}$.

The general solution is a linear combination of linearly independent vectors that span the solution space:

$$\mathbf{y} = c_1\mathbf{v_1} + ... c_n\mathbf{v_n}$$

FYI: You could see this again:

Math 4050: Solve homogeneous linear recurrance relation $x_n - x_{n-1} - x_{n-2} = 0$ where $x_1 = 1$ and $x_2 = 1$.

Fibonacci sequence: $x_n = x_{n-1} + x_{n-2}$

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

Note $x_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$

Proof: $x_n = x_{n-1} + x_{n-2}$ implies $x_n - x_{n-1} - x_{n-2} = 0$

Suppose $x_n = r^n$. Then $x_{n-1} = r^{n-1}$ and $x_{n-2} = r^{n-2}$

Then $0 = x_n - x_{n-1} - x_{n-2} = r^n - r^{n-1} - r^{n-2}$

Thus $r^{n-2}(r^2 - r - 1) = 0$.

Thus either $r = 0$ or $r = \frac{1 \pm \sqrt{1-4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$

Thus $x_n = 0$, $\quad x_n = \left(\frac{1+\sqrt{5}}{2}\right)^n$ and $\quad f_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$

are 3 different sequences that satisfy the

homog linear recurrence relation: $x_n - x_{n-1} - x_{n-2} = 0$.

Hence $x_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$ also satisfies this

homogeneous linear recurrence relation.

Suppose the initial conditions are $x_1 = 1$ and $x_2 = 1$

Then for $n = 1$: $x_1 = 1$ implies $c_1 + c_2 = 1$

For $n = 2$: $x_2 = 1$ implies $c_1 \left(\frac{1+\sqrt{5}}{2}\right) + c_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$

We can solve this for c_1 and c_2 to determine that

$$x_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$$